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# THE USE OF INTEGRAL TRANSFORMS IN THE ESTIMATION OF TIME VARIABLE PARAMETERS

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## SUMMARY

A study has been made of the concept of integral transformations applied to the differential form of the system equations and the use of these equations in parameter estimation. The two principal results of this study are: the extension of the transformation equations to explicitly account for time variability of the system parameters and an extension of the application of the concept to the output error formulation of the parameter estimation problem. It is shown that, as a result, the output error formulation acquires stability properties that were impossible previously.

Experimental results are presented to illustrate the theoretical developments. These results show the ability of the new formulations to estimate parameters that are highly variable with time.

## INTRODUCTION

The problem considered in this report can be stated, in general, as one of obtaining information about a physical system from knowledge of its input and output. A basic assumption will be made that there is sufficient understanding of the system to permit formulation of a mathematical description of the process involved. In a great majority of engineering situations this is a realistic assumption; usually enough is known about the system being investigated so that, over the range of operating conditions of interest, the numerical values of the system parameters are the primary unknowns rather than the mathematical structure of the relationships among them.

The types of parameter estimation formulation commonly called equation error and output error are considered in this report, with primary emphasis on the latter. Both formulations may utilize a predetermined mathematical description (system equation) as just mentioned, but their operation is fundamentally different. If the system output is completely related to the input through the system equation, the two schemes may be defined as follows. Equation error is formed by weighting the system variables with estimates of the system parameters and summing. If the parameter estimates equal the true process parameters, the sum will be zero; if not, the sum will equal a quantity called the equation error. Output error is formed by the difference between the output of the physical system being investigated and the output of a model excited by the same input and described by the same system equation but whose

parameters are again estimates of the true system parameters. When the estimated and true parameters are equal, the outputs of the system and model become equal, and the output error vanishes.

A significant consideration with regard to these methods relates to the variables appearing in the system equation. A very large class of physical systems can be described with acceptable accuracy by means of ordinary differential equations, and the great majority of parameter estimation work has been based on this class of system. A fundamental difficulty with this type of equation is that it is impossible generally to measure the derivatives of the system input and output appearing in it.

One solution to this problem was developed by Meissinger (ref. 1) in the form of parameter influence coefficients. These influence coefficients are the partial derivatives of system variables with respect to system parameters, and thus provide the information necessary for parameter adjustment in the output error formulation described above. This technique has been used successfully in a number of applications (refs. 1 and 2).

Another approach to the problem, the one with which this report is concerned, lies in transforming the system equation so that it is written in terms of new state variables that can be generated from those that are measurable. One of the earliest rigorous and useful transformations of this type to be applied to parameter estimation was developed by Shinbrot (ref. 3). His transformation consisted of integrals of the product of system variables and appropriate "method functions." In this way, the system differential equation was transformed into an equation with easily obtained state variables, and parameter estimates were then obtained by the equation error (then called the equations of motion) method.

Much later, but apparently independently, Zaborsky et al. (ref. 4) rederived Shinbrot's method in connection with the identification portion of an adaptive flight-control system. Again, the equation error type of parameter estimation was used. It is interesting that Zaborsky utilized a form of method function specifically excluded by Shinbrot on the basis of inaccuracy of the parameter estimates. Although some problems of this sort were experienced, the system apparently worked well, and the variables generated by the method function used were simple and required a minimum of the limited capacity of the on-board computer.

These transformations may be viewed as convolutions with the impulse response of nonphysically realizable filters. Transformations using physically realizable filters were developed almost simultaneously by three independent researchers. Valstar (ref. 5) and Rucker (ref. 6) in the United States, and Young (ref. 7) in England, although motivated differently and approaching the subject from slightly different viewpoints, arrived at essentially identical results: namely, that the system differential equation can be transformed into a more readily usable state variable form by passing the system input and output through successive physically realizable transformation filters. Each researcher utilized his results in an equation error formulation of the parameter estimation problem.

The transformation formulations developed to date are strictly valid only for constant parameter systems. The present report extends the concept by deriving transformation equations that account for time-variant parameters.

Another aspect of these transformations regards their interpretation. As noted, their developers have taken the point of view that they provide only a generalized equation error structure for parameter estimation. This point of view is also maintained in reference 8, where comparisons are made between the performance capabilities of equation error and output error systems, and a plausibility argument is given for output error systems (based essentially on Meissinger's influence coefficient approach) being stable only for sufficiently low gains. Asymptotic stability is proven for the generalized equation error system based on the formulation of Rucker. The present report shows that by taking the viewpoint that the transformed system equation represents a generalized model structure, most of the stability properties associated with the equation error formulation also apply to the output error case.

Experimental results are presented to illustrate the theory and concepts developed.

## NOTATION

| | |
|---|---|
| a,b | parameters of the plant system equation |
| c | system output |
| $E_i$ | equation error or output error |
| m | c + n |
| n | additive noise or transformation order |
| P | performance criterion |
| s | Laplace transformation variable |
| t | time |
| $T_r q$ | rth transformation of a quantity q (eq. (6)) |
| u | input |
| $v_p$ | parameter rate of change in percent of mean value per second |
| $y_i$ | ith model output |
| $\alpha_{in},\beta_{im}$ | estimate of $\frac{d^i a_n}{dt^i}, \frac{d^i b_m}{dt^i}$ |

$\tau$ transformation filter time constant, sec

$\omega$ frequency, rad/sec

$\omega_p$ frequency of harmonic parameter variation, rad/sec

$(\dot{\ }),(\ddot{\ })$ $\frac{d(\ )}{dt}$, $\frac{d^2(\ )}{dt^2}$

## THEORETICAL DEVELOPMENT AND ANALYSIS

### Transformation of a System Equation With Variable Parameters

Consider a system with input u and output c which can be described by a differential equation

$$\sum_{n=o}^{N} a_n \frac{d^n c}{dt^n} - \sum_{m=o}^{M} b_m \frac{d^m u}{dt^m} = 0 \tag{1}$$

in which there are $N + M + 1$ independent, possibly time varying, parameters. Operate on equation (1) with the integral transformation

$$\int_o^t h(t - \xi_j)d\xi_j \ . \ . \ . \int_o^{\xi_3} h(\xi_3 - \xi_2)d\xi_2$$

$$\int_o^{\xi_2}\left[\sum_{n=o}^{N} a_n(\xi_1) \frac{d^n c}{d\xi_1^n} - \sum_{m=o}^{M} b_m(\xi_1) \frac{d^m u}{d\xi_1^m}\right] h(\xi_2 - \xi_1)d\xi_1 = 0 \tag{2}$$

This transformation consists of j convolutions of equation (1) with a transformation filter defined by an impulse response h(t) which, as indicated by the integration limits, is to be that of a physically realizable system. For the purposes of this report, we will use the simplest filter and the one that has been employed most frequently, namely, a first-order lag defined by the system function

$$H(s) = \frac{1}{\tau s + 1} \tag{3}$$

or the impulse response

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \tag{4}$$

The details of carrying out the transformation (2) utilizing this impulse response are given in appendix A, where it is shown that the jth transformation of the term $g_k(d^kq/dt^k)$ is given by

$$I_j^k = \sum_{i=o}^{\infty} (-1)^i \frac{(i+j-1)!}{i!(j-1)!} \tau^{i-k} \frac{d^i g_k}{dt^i} \sum_{l=o}^{k} (-1)^l \frac{k!}{l!(k-l)!} (T_{i+j-k+l}q) + f(t) \tag{5}$$

where

$$j \geq k$$

$$0 \leq k \leq \max [N,M]$$

$$q = c \text{ or } u$$

$$g = a \text{ or } b$$

and where $f(t)$ is an exponentially decaying function of time generated by initial conditions of the parameters, the input $u$, the output $c$, and their derivatives. The symbol $T_rq$ represents the rth operation on $q$ by the transformation filter described by equations (3) and (4):

$$T_rq = \frac{1}{\tau^r} \int_o^t e^{\frac{\xi_r - t}{\tau}} d\xi_r \ . \ . \ . \int_o^{\xi_3} e^{\frac{\xi_2 - \xi_3}{\tau}} d\xi_2 \int_o^{\xi_2} q(\xi_1) e^{\frac{\xi_1 - \xi_2}{\tau}} d\xi_1 \tag{6}$$

where $\xi_{r+1} = t$ and $T_oq = q$. If now the general term (5) is used in equation (2), the result may be written, for a given order of transformation, $j \geq \max[N,M]$, as

$$\sum_{i=o}^{\infty} \left[ \sum_{n=o}^{N} A_{in}C_{in} - \sum_{m=o}^{M} B_{im}U_{im} \right] + F(t) = 0 \tag{7}$$

where

$$A_{in} = \frac{d^i a_n}{dt^i} \tag{7a}$$

$$B_{im} = \frac{d^i b_m}{dt^i} \tag{7b}$$

$$C_{in} = (-1)^i \frac{(i+j-1)!}{i!(j-1)!} \tau^{i-n} \sum_{l=o}^{n} (-1)^l \frac{n!}{l!(n-l)!} T_{i+j-n+l}c \qquad (7c)$$

$$U_{im} = (-1)^i \frac{(i+j-1)!}{i!(j-1)!} \tau^{i-m} \sum_{l=o}^{m} (-1)^l \frac{m!}{l!(m-l)!} T_{i+j-m+l}u \qquad (7d)$$

The choice of this value for $j$ eliminates all derivatives of system state variables in equation (7). If some derivatives of system state variables are measurable, they may be retained by choosing an appropriately smaller value for $j$.

Some discussion of this result is in order. By application of the transformation (2), with $h(t)$ given by equation (4), to the differential form of the system equation (1), we have arrived at equation (7), which is an exact integral form of the system equation.

The transient response of the differential form is duplicated by introduction of the forcing function $F(t)$.

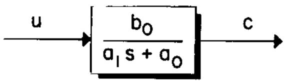
The forced response of the differential form is duplicated by appropriately filtering the input, equation (7d), to compensate for the filtering performed on the output, equation (7c). It will be seen in a subsequent example with constant parameters that the input filtering introduces zeros that just cancel poles introduced in the system itself so that the overall system function, including the input filter, matches the system function of the differential form.

Parameter variability effects are duplicated through the infinite expansion in increasing orders of parameter derivatives in equation (7).

Equation (7) does not represent a unique system description: an arbitrarily large set of system equations of the form of equation (7) may be constructed simply by setting, in equation (5), $j = \max[N,M] + i$, $i = 0, 1, 2, \ldots$, by setting $j = \max[N,M]$ and using a different value of $\tau$ for each transformed equation, or any combination of these two methods. Still another alternative exists without changing the form of the transformation filter. Equation (5) was developed for the case in which all the transformation filters in a given $T_r q$ were characterized by the same time constant $\tau$ as indicated in equation (6). Clearly, this is not a necessary restriction; each succeeding transformation filter could have a different value of $\tau$. The resulting equation corresponding to equation (5) is considerably more complex and will not be presented in this report, but there clearly are an infinity of ways of generating independent transformed system equations.

## Transformation of an Example System Equation

In this section, we will consider a simple example to illustrate the preceding development. Consider the system shown in sketch (a) described by a differential equation of first order:




Sketch (a)

$$a_1\dot{c} + a_o c - b_o u = 0 \tag{8}$$

Here, $N = 1$ and $M = 0$ and thus there are $N + M + 1 = 2$ independent parameters. Without loss of generality then, one of the parameters may be set equal to unity, but for the present all three will be retained to illustrate the transformation equations. With $j = N = 1$ in equation (7), the first transformed system equation is found to be, upon multiplying through by $\tau$,

$$\left.\begin{array}{l} a_1(c - T_1c) + a_o\tau(T_1c) - b_o\tau(T_1u) \\ - \dot{a}_1\tau(T_1c - T_2c) - \dot{a}_o\tau^2(T_2c) + \dot{b}_o\tau^2(T_2u) \\ + \ddot{a}_1\tau^2(T_2c - T_3c) + \ddot{a}_o\tau^3(T_3c) - \ddot{b}_o\tau^3(T_3u) \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ + F(t) = 0 \end{array}\right\} \tag{9}$$

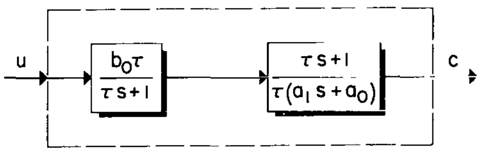
where $F(t) = -a_1(0)c(0)e^{-t/\tau}$. This equation clearly shows some of the points mentioned earlier. Consider the initial condition response. As can be seen by definition (6), the initial values of the transformed input and output variables are equal to zero (excluding impulsive-type inputs), and at time zero equation (9) reduces to the identity

$$a_1(0)c(0) - a_1(0)c(0) = 0$$

Therefore, if $u = 0$ for all time, then $T_r u = 0$ for all time, and the initial condition response is given by the response of the remaining terms in equation (9) to the forcing function $F(t) = a_1(0)c(0)e^{-t/\tau}$.

To illustrate how the forced response of the integral form of the system equation (9) duplicates that of the differential form, equation (8), consider the parameters to be fixed and take the Laplace transform of equation (9):

$$\frac{\tau(a_1 s + a_o)}{\tau s + 1}\, c(s) - \frac{b_o\tau}{\tau s + 1}\, u(s) = 0$$




Sketch (b)

In block diagram form (sketch (b)), it can be seen how the pole at $s = -1/\tau$ in the input filter just cancels the zero in the plant which has been introduced by the transformation. The overall system function thus remains the same as shown in sketch (a).

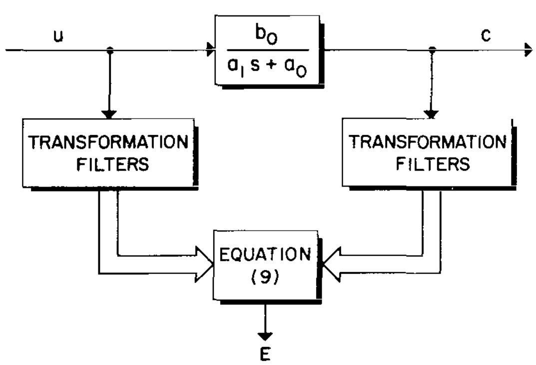
By setting $j = N + 1 = 2$ in equation (7), the second transformed system equation is found to be

$$\left.\begin{aligned}
&a_1(T_1c - T_2c) + a_0\tau(T_2c) - b_0\tau(T_2u) \\
&- \dot{a}_1 2\tau(T_2c - T_3c) - \dot{a}_0 2\tau^2(T_3c) + \dot{b}_0 2\tau^2(T_3u) \\
&+ \ddot{a}_1 3\tau^2(T_3c - T_4c) + \ddot{a}_0 3\tau^3(T_4c) - \ddot{b}_0 3\tau^3(T_4u) \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&+F(t) = 0
\end{aligned}\right\} \tag{10}$$

where now $F(t) = -a_1(0)c(0)(t/\tau)e^{-t/\tau}$. Additional system equations can be generated by further transformations, or, as mentioned earlier, by the use of various values of $\tau$ in equations (9) and (10).

## Examples of Two Parameter Estimation Formulations

The use of the transformed type of system equation in parameter estimation will be examined in detail in the following sections, but some preliminary discussion is in order here regarding the approach that has been used exclusively to date, and an extension to this approach. Consider the equation error type of parameter estimation scheme; an example of this formulation can be illustrated by equations (8) and (9) and the block diagram of sketch (c). If equation (8) defines a system whose parameters are to be numerically evaluated, and the input and output are transformed and combined as in equation (9), then if the parameters appearing in equation (9) are inaccurate estimates of the true parameters, the sum will not equal zero as shown, but will equal a quantity shown in sketch (c) as E, which is called the *equation error*. The advantage of this form of the system equation is that the variables appearing in it are easily obtained by simple integral operations on the input and output, in contrast



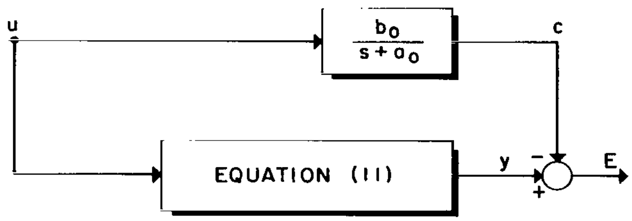

Sketch (c)

to the often impossible task of obtaining the variables of the differential form of the system equation. This, of course, was the primary motivation for the original development of the concept. An additional feature has now been added: equation (9) explicitly includes terms that account for parameter variability and thus provides more accurate modeling of time-variable systems. Although this development has been made here only for the case of a physically realizable transformation filter, an equivalent development is possible for the nonphysically realizable transformation filters discussed in the Introduction.

As mentioned in the Introduction, the originators of this type of transformed system equation, and those who have subsequently used this approach, have considered the equations only in the above context, that is, as a generalized structure for the equation error formulation of the parameter estimation problem. In the present report, we will show that another point of view is possible, namely, that the transformed system equations provide also a generalized model structure that can be used in the output error formulation of the parameter estimation problem. This can be illustrated by using equations (8) and (9). Equation (9) has already been shown to define the same dynamic structure as equation (8). Thus, it can be considered to define a model of the system defined by equation (8), a separate dynamic entity which, if its parameters and initial conditions equal those of the system, and if it is forced by the same input that forces the system, will have a response identical to that of the system. Consider the constant parameter, forced response case of equation (8) with $a_1$ set equal to unity. Designate the model output as $y$. Then equation (9) may be rewritten[1] as

$$y = T_1 y - \alpha_{oo}\tau(T_1 y) + \beta_{oo}\tau(T_1 u) \tag{11}$$

and used to define a model in an output error formulation as indicated in sketch (d). When the model parameters $\alpha_{oo}$ and $\beta_{oo}$ are inaccurate estimates of $a_o$ and $b_o$, the model output $y$, and the system output $c$ are not equal. Their difference is shown in sketch (d) as $E$, which is called the output error. The next section will show that here also, as in the case of the equation error formulation, there are significant advantages to using the integral form of the system equation rather than the differential form.




Sketch (d)

[1]The double subscript notation introduced here for the model parameters will be used to designate both the appropriate system parameter and its particular derivative to which the model parameter corresponds. For instance, $\alpha_{in}$ corresponds to $d^i a_n/dt^i$.

## Parameter Adjustment

To this point, all that has been said regarding the parameter estimates in either the equation error or output error formulation is that if they are equal to the true parameters, the resultant errors (E in sketches (c) and (d)) vanish. The means by which the parameter estimates are to achieve the correct values has not been described.

Two basic approaches exist, each of which is uniquely suited to a specific type of computer. The first is matrix inversion. In the preceding section two independent parameters exist in the system described by equation (8). If the parameters are constant then equations (9) and (10) reduce to the time-invariant case given by the first line, and a matrix inversion solution yields the two values. This type of solution obviously requires a digital computer.

Solution by analog computer may be obtained by gradient-type techniques. A nonnegative, increasing performance criterion must be selected, for instance,

$$P = \frac{1}{2} E^2 \tag{12}$$

A parameter adjustment strategy that makes use of knowledge of the gradient of the surface formed by the performance criterion must also be chosen. One strategy commonly used is that of steepest descent (ref. 9), defined, for the case of continuous parameter adjustment, by

$$\dot{\gamma}_{ij}(t) = -k_{ij} \frac{\partial P(t)}{\partial \gamma_{ij}(t)} \tag{13}$$

where $\gamma_{ij}$ represents each of the independent parameter estimates ($\alpha_{ij}$ and $\beta_{ij}$) in turn. Consider the meaning of equation (13). The rate $\dot{\gamma}_{ij}$ at which the parameter estimate is varied at time $t$ is proportional to the negative change of $P$ at time $t$ per unit change of $\gamma_{ij}$ at time $t$. In the equation error formulation, the instantaneous gradient component $\partial P(t)/\partial \gamma_{ij}(t)$ of the surface formed by the performance criterion is given directly by the state variable associated with the parameter being considered. This is true for both the differential and integral forms of the system equation.

In contrast with the availability of exact instantaneous gradient information obtainable in the equation error formulation, continuous parameter adjustment in the output error formulation must use inexact information when the differential form of the system equation is used. This is due to the fact that the instantaneous gradient component $\partial P(t)/\partial \gamma_{ij}(t)$ is identically equal to zero since the model output does not respond instantaneously to changes in parameter estimates. The parameter estimates must, therefore, be adjusted in accordance with a prediction of the future response of the model output due to

a change in the parameter estimate "now" at time $t$. This is the type of information provided by Meissinger's influence coefficient solutions. However, the underlying theory is strictly valid only for zero rate of adjustment of the parameter estimates (ref. 2), and the accuracy of the information degenerates as the adjustment rate increases. This limits the use of techniques based on Meissinger's influence coefficients to parameter estimation of stationary or very slowly varying systems.

Use of a model defined by the integral form of the system equation eliminates this difficulty entirely; the model output now responds instantaneously to changes in the parameter estimates, the instantaneous gradient components are again nonzero, and their values are obtained in the same straightforward manner as for the equation error formulation.

## Stability

Many proofs can be found in the literature that the equation error formulation of the parameter estimation problem is stable (see, e.g., ref. 8). The proofs remain valid when the parameter variability effects are explicitly included in the transformed system equation. No such proofs can be found for the output error formulations that utilize continuous parameter adjustment. To date, only the differential form of the system equation has been used, and gradient information has been obtained from methods essentially the same as that of Meissinger (refs. 8 and 10). Although successful convergence of the parameter estimates has been achieved with appropriate choices for the parameter adjustment rate gains ($k_{ij}$ in eq. (13)), experience seems to indicate such formulations can always be made unstable by choosing sufficiently large gain values (refs. 2 and 8). The stability of the output error formulation, when a model defined by the integral form of the system equation is used, will be examined in this section.

Consider a system defined by the differential equation (1). Initial conditions have no place in the following discussion, so they may be assumed to equal zero. Then the system is also defined by equation (7) with $F(t) = 0$. Construct a model of the system

$$\sum_{i=o}^{\infty}\left[\sum_{n=o}^{N}\alpha_{in}Y_{in} - \sum_{m=o}^{M}\beta_{im}U_{im}\right] = 0 \tag{14}$$

and normalize the system equation (7) by the $n = N$ zeroth derivative parameter. Normalize equation (14) by the corresponding parameter estimate. Then

$$A_{ON} = \alpha_{ON} = 1 \tag{15}$$

It immediately follows that $A_{iN} = \alpha_{iN} = 0$ for $i > 0$. If we use the performance criterion (12), and define the error in a manner similar to

sketch (d), then[2]

$$E = Y_{ON} - C_{ON} \tag{16}$$

Adjusting the model parameter estimates according to equation (13) gives

$$\left.\begin{aligned} \dot{\alpha}_{in} &= +k_{in}EY_{in}\,, \qquad n \neq N \\ \dot{\beta}_{im} &= -k_{im}EU_{im} \end{aligned}\right\} \tag{17}$$

Now the performance criterion will vary with time according to

$$\begin{aligned} \dot{P} &= E\dot{E} \\ &= E\sum_{i=o}^{\infty}\left[\sum_{n=o}^{N-1} - Y_{in}\dot{\alpha}_{in} + \sum_{m=o}^{M} U_{im}\dot{\beta}_{im}\right] + E\left(\frac{\partial E}{\partial t}\right)_{\alpha,\beta} \end{aligned} \tag{18}$$

Substituting equation (17) in (18) gives

$$\dot{P} = -E^2\sum_{i=o}^{\infty}\left[\sum_{n=o}^{N-1} k_{in}Y_{in}^2 + \sum_{m=o}^{M} k_{im}U_{im}^2\right] + E\left(\frac{\partial E}{\partial t}\right)_{\alpha,\beta} \tag{19}$$

The last term in equation (19) is a function of system and model input, parameter estimate inaccuracies, and system parameter variability. The first (bracketed) term is due to adjustment of the parameter estimates in the model. Thus, the effect of adjusting the parameter estimates according to equation (17) is to drive the performance criterion toward zero, and a zero value will be assured if

$$E^2\sum_{i=o}^{\infty}\left[\sum_{n=o}^{N-1} k_{in}Y_{in}^2 + \sum_{m=o}^{M} k_{im}U_{im}^2\right] > E\left(\frac{\partial E}{\partial t}\right)_{\alpha,\beta} \tag{20}$$

a condition which can be realized with sufficiently large adjustment gains.

Satisfaction of inequality (20) guarantees convergence of the performance criterion to zero; it does not, however, guarantee that the parameters will

[2]Sketch (d) defines $E = y - c$. In terms of the specific example considered previously, equations (9) and (11), the definition (16) gives $E = (y - T_1y) - (c - T_1c)$. This definition simplifies the succeeding equations, and has no effect on the arguments presented, which apply equally well to the definition of sketch (d).

converge to a unique set. It is obvious from equation (14), the model system equation, that at any instant of time there are an infinity of combinations of parameter estimates which will satisfy $P = 0$. This result has been thoroughly discussed in the literature (refs. 2 and 8), and is due to the fact that the performance criterion, equation (12), does not represent a surface with closed contours of constant $P$. The analogous digital solution difficulty is a singular matrix resulting from more unknowns than equations. Unlike the digital case, however, convergence to a unique solution is still possible in the analog gradient technique case if the adjustment gains for the parameter estimates are sufficiently small (refs. 2 and 8).

Uniqueness can be guaranteed by generating a set of models, by one or more of the means previously described, equal to or greater in number than the parameters to be evaluated. An error vector can then be defined with components

$$E_r = Y_{ONr} - C_{ONr}, \qquad r = 1,2, \ldots, R \tag{21}$$

and used in a performance criterion

$$P = \frac{1}{2}\sum_{r=1}^{R} E_r^2 \tag{22}$$

Parameter estimate adjustment now proceeds as

$$\left.\begin{aligned} \dot{\alpha}_{in} &= k_{in}\sum_{r=1}^{R} E_r Y_{inr}, \qquad n \neq N \\ \dot{\beta}_{im} &= -k_{im}\sum_{r=1}^{R} E_r U_{imr} \end{aligned}\right\} \tag{23}$$

The variation with time of the performance criterion (22) for this case is

$$\left.\begin{aligned} \dot{P} = -\sum_{i=o}^{\infty}\Bigg\{&\sum_{n=o}^{N-1} k_{in}\left[\sum_{r=1}^{R} E_r Y_{inr}\right]^2 \\ &+ \sum_{m=o}^{M} k_{im}\left[\sum_{r=1}^{R} E_r U_{imr}\right]^2\Bigg\} + \sum_{r=1}^{R} E_r\left(\frac{\partial E_r}{\partial t}\right)_{\alpha,\beta} \end{aligned}\right\} \tag{24}$$

and again it can be seen that minimization of the performance criterion is assured with sufficiently large gain values. Now, however, since the performance criterion is formed using independent models that, in number, are equal to or greater than the number of parameters, the set of parameter estimates which minimizes P is unique.

Thus, stability of the output error formulation is assured when models defined by the integral form of the system equation are used, and uniqueness of the parameter estimates can be assured by utilizing a sufficient number of models. The reader familiar with the equation error formulation will recognize the direct parallel of these results to that case, but will note also a significant difference. Whereas, in the equation error formulation, minimization of the performance criterion combined with uniqueness of the parameter estimates means that the parameter estimates have achieved the values of the system parameters, this is not necessarily true for the output error formulation. Satisfaction of the two conditions guarantees that the parameter estimates will ultimately converge to the correct values, but instead of achieving the correct values coincident with minimization of the performance criterion as in the equation error formulation, the analysis in the next section will show the correct values will be achieved some time after minimization has occurred.

## Convergence of the Parameter Estimates - Output Error Formulation

In this section, convergence of the parameter estimates will be analyzed for a particular example. The example will also serve to illustrate the structure of the output error formulation when a vector error is used. The first-order system defined by equation (8) will again be used. Equation (9) is an integral form of the system equation, the form that will be used to define the models. Assume the two independent unknown parameters as $a_o$ and $b_o$. The parameter $a_1$ may be set equal to unity and, of course, all its derivatives equal to zero. For simplicity, and with no loss of generality as regards time for parameter convergence, assume also that $b_o$ is time invariant.

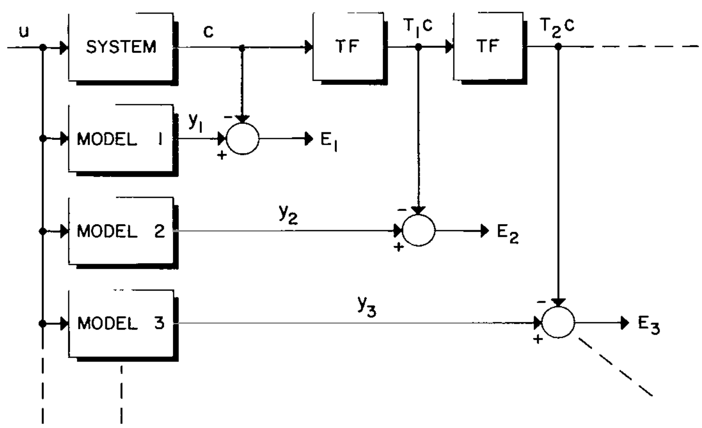
Assume zero initial conditions. Then equation (9) may be written

$$c - T_1 c + a_o \tau T_1 c - \dot{a}_o \tau^2 T_2 c + \ddot{a}_o \tau^3 T_3 c - \ . \ . \ . \ . - b_o \tau T_1 u = 0 \qquad (25)$$

The number of unknown parameters in this equation is determined by the number of derivatives of $a_o$ necessary to adequately approximate its variability. If the vector error, equation (21), is used to ensure parameter estimate uniqueness, then the minimum necessary number of error components (and, therefore, system models) is determined by the number of unknown parameters in equation (25). Each model must be defined by an equation that corresponds to an independent system equation, formed by one or more of the methods described previously. If, in the present example, these equations are formed by setting $j = 1,2,3, \ . \ . \ .$ in equation (5), then equations (9), (10), and subsequent transformed system equations result. With the conditions that

enable equation (9) to be written as equation (25), equation (10) may be written as

$$T_1c - T_2c + a_o\tau T_2c - \dot{a}_o 2\tau^2 T_3c + \ddot{a}_o 3\tau^3 T_4c - \dots - b_o\tau T_2u = 0 \qquad (26)$$




Sketch (e)

and so on for the higher transformations. The output of the system defined by equation (25) is $c$, that of the system defined by equation (26) is $T_1c$, that defined by the higher transformations, $T_2c$, $T_3c$, etc. With a model defined to correspond to each of these variables, the structure of the output error formulation appears as shown in sketch (e), where TF denotes the transformation filter defined by equation (3) or (4).

The first model, whose output is to correspond to equation (25), is defined by

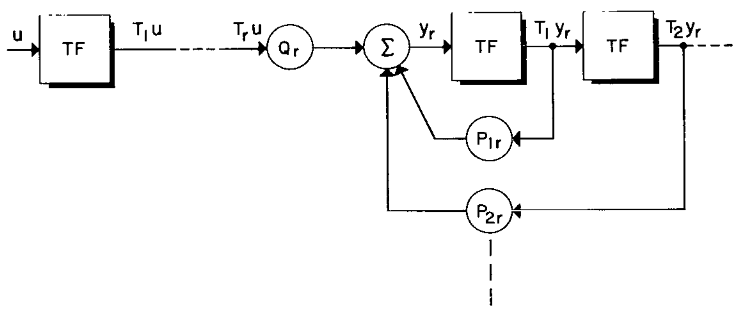
$$y_1 - T_1y_1 + \alpha_{oo}\tau T_1y_1 - \alpha_{1o}\tau^2 T_2y_1 + \alpha_{2o}\tau^3 T_3y_1 - \dots - \beta_{oo}\tau T_1u = 0 \qquad (27)$$

The second model is defined by

$$y_2 - T_1y_2 + \alpha_{oo}\tau T_1y_2 - \alpha_{1o}2\tau^2 T_2y_2 + \alpha_{2o}3\tau^3 T_3y_2 - \dots - \beta_{oo}\tau T_2u = 0 \qquad (28)$$

and, similarly, for the models corresponding to the higher transformations. In general, the model equations can be written

$$y_r = P_{1r}T_1y_r + P_{2r}T_2y_r + P_{3r}T_3y_r \dots + Q_rT_ru \qquad (29)$$




Sketch (f)

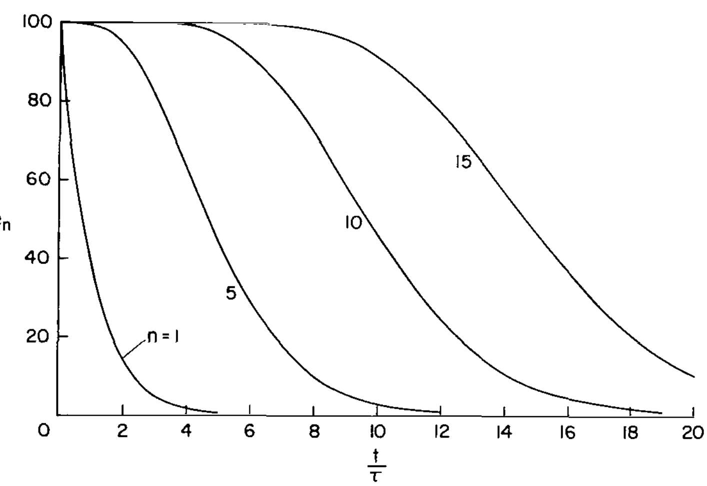
where the $P_{ir}$ and $Q_r$ contain the parameter estimates appropriately weighted for each model.

The important point here is the model structure implied by equation (29), shown in sketch (f). Consider that the system and model variables have all reached their steady-state condition (i.e., that all transients have disappeared) and

that the parameter estimates are set at some incorrect value. Then the error vector has a nonzero value. If now the parameter estimates are adjusted in accord with equations (23), and if the adjustment gains are sufficiently large, the error vector can be driven quickly to the vicinity of zero and constrained to remain there. As discussed previously, although the set of parameter estimates that initially forces the errors to the vicinity of zero are unique, they cannot equal the true parameters. This is because the model structure of sketch (f) defines also the system structure, and the parameter estimates cannot equal the true parameters until the corresponding variables in model and system are equal. This is, in model 1, $T_1y_1$, $T_2y_1$, . . . must equal $T_1c$, $T_2c$, . . .; in model 2, $T_1y_2$, $T_2y_2$, . . . must equal $T_2c$, $T_3c$, . . .; and so on. In general $T_ny_r$ must equal the system variable $T_{n+r-1}c$ in order that the parameter estimates equal the true parameter values. The time it takes $T_ny_r$ to equal $T_{n+r-1}c$, if $E_r$ is forced instantaneously to zero, can be estimated from the propagation time of a step input through the series of transformation filters in a model as shown in sketch (f). The percent error between the two variables is given by

$$e_n = 100\left(\frac{T_{n+r-1}c - T_ny_r}{T_{n+r-1}c}\right)$$

$$= 100\left[1 + \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2 + \ . \ . \ . \ + \frac{1}{(n-1)!}\left(\frac{t}{\tau}\right)^{n-1}\right]e^{-t/\tau} \tag{30}$$



Sketch (g)

and is shown in sketch (g). From this relationship, the time for effective parameter convergence can be estimated from the time for the difference between the highest order transformation variables to effectively disappear. Figure 1 gives the time for the error to drop to a value of 1 percent.

This is a general result since sketch (f), to the right of the summer symbol, is a general picture of model structure, the number of transformations depending on the number of system poles and the variability of the $a_n$ parameters. The number of input transformations to the left of the summer symbol is determined by the variability of the $b_m$ parameters and by the number of system zeros, but there is no effect on the length of time for parameter convergence since the input transformations are not affected by changes in the parameter estimates.

Although finite parameter adjustment gains prevent the error vector from being driven to zero instantaneously as was assumed here, it will be seen subsequently in the experimental results that the parameter convergence times can be estimated with fair accuracy using figure 1.

### Estimation of Nonlinear Parameters

In the preceding sections, the parameters of the system equation have been considered to be time variable and, at least implicitly, independent of the system variables. This viewpoint is not necessary. For instance, in the first-order system example considered in the foregoing section, the parameter $a_o$ could be considered to be a time-invariant nonlinear function of the system output $c$. It immediately follows that it is then expressible as a time-varying quantity, and can be estimated by the means just discussed. Recovery of the nonlinear function would then be achieved from a plot of the parameter estimate $\alpha_{oo}$ versus the system output.

Another approach that can be used with the present transformation equations is due to Shinbrot (ref. 3). For example, if equation (8) is written as

$$\dot{c} + a_o(c)c - b_o u = 0 \tag{31}$$

the parameter $a_o(c)$ is expressed as a polynomial

$$a_o(c) = k_o + k_1 c + k_2 c^2 + \ldots \tag{32}$$

of the complexity felt to be required for adequate approximation. Substituting equation (32) in (31) and applying the transformation equations gives, for the first transformation,

$$c - T_1 c + k_o \tau T_1 c + k_1 \tau T_1(c^2) + k_2 \tau T_1(c^3) + \ldots - b_o \tau T_1 u = 0 \tag{33}$$

and so on as before for the following transformations. Parameter estimation proceeds exactly as for the linear case.

There are no conceptual difficulties in generalizing further to systems with time-varying nonlinear parameters. The coefficients in the polynomial (32) are then functions of time, and are handled in the manner already illustrated.

## PRACTICAL CONSIDERATIONS

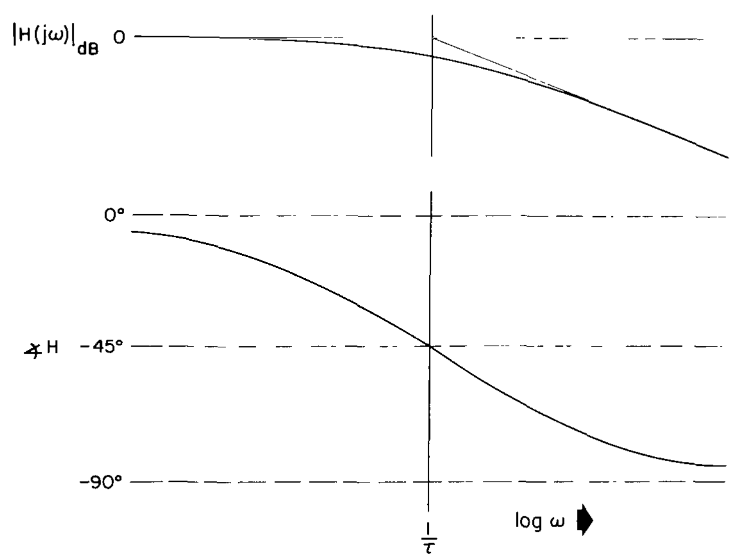
As a result of the development and analyses of the preceding sections, it is possible to draw certain inferences regarding the practical use of the transformed system equations in the parameter estimation problem.

## Transformation Filter Time Constant

The transformation filter time constant, $\tau$, is fundamental to all the results developed in the preceding sections, and thus, perhaps not unexpectedly, satisfaction of various criteria can put conflicting requirements on the numerical value to be chosen. For instance, rapid convergence of the parameter estimates to the true parameter values is a very desirable feature. The foregoing section has shown that, in the output error formulation, convergence time is directly expressible in terms of $\tau$, and thus $\tau$ should be as small as possible.

A somewhat parallel consideration relates to parameter variability. If all orders of derivatives are generated by the parameter variability, then obviously the integral form of the system equation must be truncated, with a resulting loss of accuracy in the system description. It can be seen (e.g., eq. (9)) that the accuracy loss can be reduced by choosing a small value for $\tau$, thus reducing the effects of the neglected higher order parameter derivatives.

In direct conflict with these requirements are the requirements arising from use of equation (21). Definition of an error vector requires models defined by independent equations, such as equation (27) and following, whose outputs correspond to variables generated by independent system equations such as equation (25) and following. The key word here is independent. As $\tau$ approaches zero, equation (25) approaches $T_1c = c$, equation (26) approaches $T_2c = T_1c$, etc. That is, the system equations are no longer independent. This is a result of the nature of the impulse response of the transformation filter, equation (4), as $\tau$ approaches zero. Regardless of the value of $\tau$, the integral of equation (4) over all positive time is unity $\sec^{-1}$, and hence equation (4), as $\tau \to 0$, is a valid definition of a unit impulse. Due to its sifting property, convolution of the unit impulse with a function simply yields the function (ref. 11). In terms of the transformation filter frequency response implied by the system function, equation (3), and as shown in sketch (h), a value of $\tau$ approaching zero implies a cutoff frequency approaching infinity and a phase shift approaching zero.



Sketch (h)

Thus, $\tau$ cannot be allowed to become too small or the information generated by sequential transformations will be so small as to be lost in the noise level inherent in any computer, and accurate parameter estimation will be impossible. Sketch (h) indicates that maximum independence of the system equations will occur with a small value of $1/\tau$ relative to the maximum frequency of the input, since each

transformation will then yield a large phase shift when viewed in terms of the frequency domain. However, a small value of $1/\tau$ relative to maximum input frequency means that considerable attenuation accompanies each operation by a transformation filter, and again accurate parameter estimation becomes difficult because of the resulting low signal levels.

Thus, the value of $\tau$ should be chosen as small as possible for fast parameter convergence, for estimation of variable parameters and to prevent loss of accuracy through excessive signal attenuation, but large enough to ensure independence of the transformed system equations. It is clear that the range of values of $\tau$ satisfying these requirements shrinks as the number of transformations required increases, and thus there is some definite practical upper limit to the complexity of the system to be investigated and to the complexity of the estimation formulation for which accurate parameter estimation is possible.

## Simplification of Output Error Formulation

The number of transformation filters it is necessary to mechanize in the output error formulation can be reduced in the following way. It was shown in the section on parameter convergence time that, after convergence, the relationship between the model and system variables is $T_n y_r = T_{n+r-1} c$. This means that, after convergence, the relationships among the model variables are $T_n y_r = T_{n-i} y_{r+i}$, $i = 1,2, \ldots, n$. If this relationship among model variables is preserved for all time, then, for instance, equation (27) may be written

$$y_1 - y_2 + \alpha_{00}\tau y_2 - \alpha_{10}\tau^2 y_3 + \alpha_{20}\tau^3 y_4 - \cdot \cdot \cdot - \beta_{00}\tau(T_1 u) = 0$$

Equation (28) becomes

$$y_2 - y_3 + \alpha_{00}\tau y_3 - \alpha_{10} 2\tau^2 y_4 + \alpha_{20} 3\tau^3 y_5 - \cdot \cdot \cdot - \beta_{00}\tau(T_2 u) = 0$$

and so on. The last model must still be formed in the original way as given by equation (29). Thus, the transformation filters in the feedback of all but the last model may be eliminated. The only effect of this formulation is the elimination of the independent transients the models previously were capable of exhibiting. The only transient now existing is due to the last model; convergence time is unaffected, however, since convergence times for all models were equal.

# EXPERIMENTAL RESULTS

## First-Order System

Experimental results for the first-order system used to illustrate the discussion of the previous section will now be presented. The system is defined by equation (8), and the two independent parameters were taken to be $a_o$ and $b_o$. The output error formulation utilizing a three component vector error was used as indicated in sketch (e). Mechanization was by analog

computer, based on equations (22) and (23). The model equations used were

$$\left.\begin{aligned} y_1 &= T_1y_1 - \alpha_{oo}\tau T_1y_1 + \beta_{oo}\tau T_1u \\ y_2 &= T_1y_2 - \alpha_{oo}\tau T_1y_2 + \beta_{oo}\tau T_2u \\ y_3 &= T_1y_3 - \alpha_{oo}\tau T_1y_3 + \beta_{oo}\tau T_3u \end{aligned}\right\} \tag{34}$$

The value of $\tau$ chosen was 0.1 second. This choice was made as small as possible on the basis of the preceding discussions. The input chosen was the sum of sine waves in all cases, three for the analog results to be presented and eight for the digital results. This type of input is simple to generate, it provides the capability for repeatable results, and it is easily made to satisfy the two principal requirements of sufficient information content and adequate system excitation. A discussion of input requirements may be found in references 2, 4, and 8.

These models and those used in subsequent examples of the output error formulation do not contain the terms corresponding to parameter derivatives which have been included in the previous theoretical developments because it was found that inclusion of these terms caused a loss of accuracy believed to be due to numerical difficulties associated with the added complexity. Inclusion of the parameter derivative terms in a digital mechanization of a subsequent example of the equation error formulation will be seen to provide more accurate estimates.

Time-variable parameter estimation.- Figure 2(a) shows the response of the parameter estimates to step changes of two different magnitudes in the system parameters. Convergence of the parameter estimates to the true values is rapid, of the order of 1 second. This value of convergence time may be checked against the value estimated using figure 1. The highest transformation appearing in a model feedback loop, from equations (34), is first order, and from figure 1, the convergence time is estimated to be approximately five transformation filter time constants, or 0.5 second. It will be recalled that this estimate is based on instantaneous reduction of the error vector to zero. The characteristics of the error traces in figure 2(a) indicate that the data were obtained with a rather low value of adjustment rate gain for the parameter estimates, and that the resulting adjustment rates were the limiting factor in the time for convergence of the parameter estimates.

Figure 2(b) shows the estimation of system parameters which are varying in a ramp fashion at rates of change $v_p$ = 5, 20, and 40 percent of the mean value per second. For this type of variation, of course, only the first parameter derivative has a nonzero value except at those unique instants of time of transition when all derivatives exist as successively higher order impulses. Since the model structure contains no variables to account for system parameter variability, one would expect the parameter estimates to be less accurate in the vicinity of the transition points. This is borne out in figure 2(b) for $v_p$ = 40 percent of the nominal value per second; in the vicinity of the first transition point, errors relative to the nominal value of

approximately 9 percent in $\alpha_{oo}$ and 18 percent in $\beta_{oo}$ occur. At the lower variability rates, the effect of the transition points is scarcely evident, and the general error level is quite low.

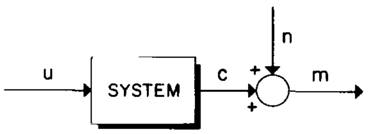
When the system parameters vary in a sinusoidal fashion, figure 2(c), all parameter derivatives exist and, for a parameter variation frequency of $\omega_p = 1$ rad/sec, the maximum values of all derivatives are equal. With a transformation filter time constant equal to 0.1, however, equation (25) shows that the effect of each succeeding parameter derivative on the output is reduced approximately an order of magnitude. Figure 2(c) shows the result: accurate parameter estimation at $\omega_p = 0.25$ and 0.5, and maximum errors of approximately 10 percent of the nominal value for $\alpha_{oo}$ and 15 percent for $\beta_{oo}$ at $\omega_p = 1$.

## Second-Order System

We present here results for a system with twice the complexity of the foregoing, a second-order system with one zero, defined by

$$\ddot{c} + a_1\dot{c} + a_o c = b_1\dot{u} + b_o u \tag{35}$$

This example will also be used to illustrate the effect of additive noise on parameter estimation, a topic which has not been mentioned to this point. The nomenclature to be used is illustrated in sketch (i), where the variable n, the noise, is added to the system output c defined by equation (35) to form the quantity m. It is assumed that only u and m are measurable, and thus the transformed system equations are given by




Sketch (i)

$$T_{i-1}m - 2T_i m + T_{i+1}m + a_1\tau(T_i m - T_{i+1}m) + a_o\tau^2(T_{i+1}m)$$

$$- b_1\tau(T_i u - T_{i+1}u) - b_o\tau^2(T_{i+1}u) = 0 \tag{36}$$

where $i = 1, 2, 3$, and 4.

The first parameter estimates to be presented were again obtained by analog computer mechanization of the steepest descent solutions of the output error formulation. Four models were used, corresponding to equation (36),

$$y_i = 2T_1 y_i - T_2 y_i - \alpha_{o1}\tau(T_1 y_i - T_2 y_i) - \alpha_{oo}\tau^2(T_2 y_i)$$

$$+ \beta_{o1}\tau(T_i u - T_{i+1}u) + \beta_{oo}\tau^2(T_{i+1}u) \tag{37}$$

where $i = 1, 2, 3$, and 4.

These models were used to form a four-component error vector (sketch (e)), except that the system variables were considered to be $T_r m$ rather than $T_r c$. It should be noted that if $n$ is simply a constant (bias), its effects can be eliminated by inclusion of another model or, perhaps more straightforwardly, by passing both $u$ and $m$ through identical high-pass filters before parameter estimation is attempted. Results will be presented first for $n = 0$.

Time-variable parameter estimation.- Experimental results for the system parameters, again performing step, ramp, and harmonic variations, are presented in figure 3, where now the true and estimated parameters are superimposed in the lower part of the figure. Figure 3(a) shows the response of the parameter estimates to step changes in the system parameters of three different magnitudes. The highest transformation in the feedback loop of each model, from equation (37), was second order. The transformation filter time constant was again set at 0.1, thus leading to an estimated convergence time from figure 1 of approximately 0.7 second. Figure 3(a) shows that, as the starting transient disappears, parameter estimate convergence time approaches this estimated value, and that convergence time is independent of parameter step size.

Figure 3(b) shows that the parameter estimates were able to follow ramp changes of the system parameters with an accuracy somewhat less than that for the first-order system, but which was nevertheless quite good. The primary error appears as an almost constant lag of the estimates behind the true parameters. Again, the effects of the higher parameter derivatives at the transition points are discernible only at the highest rate shown.

The results for the harmonic parameter variation presented in figure 3(c) also show some reduced accuracy relative to the first-order system. In this figure, it can be seen that the primary cause of inaccuracy was a slight phase shift between the true and estimated parameters; maximum excursion amplitudes were estimated fairly accurately. Even at $\omega_p = 2$ rad/sec, where periodically the error reached values exceeding 50 percent of nominal for $\beta_{00}$, it can be seen that a fairly accurate overall picture of parameter variability is obtained.

Error vector components.- As discussed earlier, parameter estimate uniqueness can be guaranteed only if the number of components in the error vector is equal to or greater than the number of parameters. It was also mentioned that, with fewer error components than parameters, parameter estimate uniqueness was still possible in the analog gradient solution case if the adjustment gains on the parameter estimates are sufficiently small.

These remarks obviously refer to the time-invariant case. The question then arises: To what extent is it possible to estimate time-variable parameters with fewer error components than parameters? Figure 4 is a partial answer to this question for the second-order system. As shown in the center of the figure, the number of error components was varied from four to one. The parameter estimate adjustment gain was maintained at the same value used to obtain the results of figure 3. It can be seen that, for this level of

gain, a slight deterioration in accuracy occurred as the number of error components was reduced to three and then to two. The scalar error case is completely unacceptable, however, with $\beta_{00}$ approximately 180° out of phase with $b_0$, and extreme errors in each of the other parameter estimates also. These results show that the estimation of parameters with a rather high degree of variability does not necessarily require a complete error vector.

Interaction of parameter estimates.- In all the preceding results, each of the system parameters was varying in the same manner - step, ramp, or sinusoid. It is therefore not apparent from these results to what extent variation of one parameter affects the estimation of another parameter.

If the system variability could be modeled exactly, and if the adjustment gain could be made sufficiently large that the error vector could be constrained to zero, no interaction between parameter estimates would be expected. The results presented so far show that sufficient gain to maintain the error vector in the close proximity of zero can indeed be achieved. System variability is not modeled by equations (37), however, and so inaccuracies in the estimate of a varying parameter must necessarily exist, with consequent inaccuracies in the other estimates also.

Figure 5 shows the magnitude of this effect in the second-order system, again for a sinusoidal variation with $\omega_p = 1$ rad/sec. The number of varying system parameters is clearly evident from the lower part of the figure, ranging from all four, which is a repeat of previous data, to a single varying parameter, $b_1$. The results show that the error in each parameter estimate is roughly independent of the number of system parameters which are varying; that is, in this case at least, the inaccuracy in a parameter estimate does not increase as the number of varying system parameters increases. This may not be a general result, but it is an encouraging one.

Adjustment gain.- Accurate estimation of variable parameters is obviously impossible if the maximum adjustment rate of the estimates is less than the maximum rate of parameter change. This determines the lower gain limit. An approach to the lower gain limit has already been indicated in the first-order system results shown in figure 2(a), where the gain was sufficiently low that the error components were allowed to achieve rather large values. Nevertheless, accurate estimation of the variable parameters was possible in figures 2(b) and 2(c).

The upper gain limit is somewhat less well defined than the lower limit. It was discussed previously how output error formulations that utilize the differential form of the system equation can always be made unstable by choosing sufficiently large parameter adjustment gains. It was also shown that use of the integral form of the system equation eliminated this type of problem. Even with this constraint on the adjustment gain removed, it will be seen that there is some maximum value of gain for the best estimation of time-varying parameters. This is indicated in figure 6 where results for the $\omega_p = 1$ rad/sec sinusoidal variation are shown for three values of gain: the nominal value used to obtain all the previous second-order results, twice the nominal, and ten times the nominal. (The second component $E_2$ inadvertently was not

recorded for the nominal case, but it actually was an active component of the error vector.) The results for the nominal and twice nominal gain values are not significantly different. The ten times nominal results show a greatly increased variance about the true parameter values, however, and thus this gain value may be considered too great.

Two effects are present which together determine the upper gain limit. The first is the noise level of the computer; increased gain causes the parameter estimates to respond to erroneous signals that are not directly related to the estimation problem. The second effect is caused by the inexact modeling of the system variability. To illustrate this, consider again the first-order system of equation (8). Let $a_o$ be the single variable parameter and $b_o$ equal unity. The system is thus described by

$$\dot{c} + a_o c = u$$

Equation (A4) can be used to write an exact integral form as

$$c - T_1 c + a_o \tau T_1 c - \tau^2 T_1(\dot{a}_o T_1 c) - \tau T_1 u = 0 \tag{38}$$

For this very simple case, the output error formulation requires but a single model. This is defined by

$$y_1 - T_1 y_1 + \alpha_{oo} \tau T_1 y_1 - \tau T_1 u = 0 \tag{39}$$

Now if a matrix inversion technique is used, or if a gradient technique is used with the adjustment gain sufficiently large, then the single required error

$$E = y_1 - c \tag{40}$$

can be held to zero. The model variables will then become equal to the system variables, and equation (39) can be written

$$c - T_1 c + \alpha_{oo} \tau T_1 c - \tau T_1 u = 0 \tag{41}$$

The value of $\alpha_{oo}$ that holds $E$ equal to zero can thus be determined from equation (41). It can be expressed in terms of the true parameter $a_o$ by subtracting equation (38) from (41) and solving to give

$$\alpha_{oo} = a_o - \tau \frac{T_1(\dot{a}_o T_1 c)}{T_1 c} \tag{42}$$

Now the zero crossings of $T_1(\dot{a}_o T_1 c)$ and $T_1 c$ obviously do not occur at the same instants of time, and thus the parameter estimate $\alpha_{oo}$ will have an infinite variation about the true (variable) value $a_o$.

The result is a consequence of infinite adjustment gain. For physically realizable gain values, the error is not held identically to zero, and the parameter estimate variability is filtered by the single integration in the

adjustment loop to give the resultant variability evidenced by all the estimates that have been presented. As the gain level is increased as in figure 6, the error components are constrained closer to zero, with the resulting increase in variance shown. There is thus no clear-cut upper gain limit, but best results will obviously be obtained by making the gain only sufficiently large to encompass the bandwidth of the variable parameters.

Equation error formulation.- We present in this section parameter estimates for the second-order system obtained by digital matrix inversion solution of the equation error formulation. The transformed system equations expressed as components in the vector equation error are given by (if all parameter derivatives higher than the first are assumed to be zero).

$$\begin{aligned} E_i = {} & T_{i-1}m - 2T_im + T_{i+1}m \\ & + \alpha_{01}\tau(T_im - T_{i+1}m) + \alpha_{00}\tau^2(T_{i+1}m) - \beta_{01}\tau(T_iu - T_{i+1}u) - \beta_{00}\tau^2(T_{i+1}u) \\ & - 2\alpha_{11}\tau^2(T_{i+1}m - T_{i+2}m) - 2\alpha_{10}\tau^3(T_{i+2}m) + 2\beta_{11}\tau^2(T_{i+1}u - T_{i+2}u) + 2\beta_{10}\tau^3(T_{i+2}u) \end{aligned} \tag{43}$$

Two sets of results will be presented, one set for which the parameter derivative terms were assumed zero, and for which $i = 1, 2, 3, 4$, and another for which the full equation (43) was used, with $i = 1, 2, \ldots, 8$.

These results are shown in figure 7 for a ramp variation of the system parameters. Large variations of the estimates about the true value of the parameters occur when the system parameters are assumed to be invariant. Addition of the parameter derivative estimates greatly improves the accuracy of the solution because of the more accurate modeling of the variable system dynamics. Maximum errors occur near the transition points where all orders of parameter derivatives exist.

The analog results presented previously (fig. 3(b), $v_p = 40$) contained the assumption of constant system parameters, yet did not exhibit the extreme variability shown by the analogous digital results. This is because of the smoothing action of the steepest descent type of solution at the gain levels used. As discussed in the foregoing section, increasing the adjustment gain tends to make the analog results increase in variability about the true value.

It will be noted that the roughly constant lag exhibited by the analog estimates of figure 3(b) has been eliminated by including the parameter derivative estimates in the digital solution. The fact that the parameter estimates in the digital solution are not exact during the ramp variation is due to computation inaccuracies introduced by a relatively large integration step size.

Noise.- The effects of additive noise on the accuracy of parameter estimation is shown in figure 8, where the noise-to-output-signal ratio (rms), n/c, is varied from 7 to 28 percent. Parameter estimate response with no additive noise is also presented for comparison. The noise was wideband -

essentially white over the frequency range of interest - filtered by a first-order low-pass filter with a breakpoint at 10 rad/sec. The sample statistics for a period of 100 seconds were as shown in the following table. The bias for all parameter estimates over the same sample period was less than 3.5 percent of their nominal values. The unbiased nature of the parameter estimates is due to the fact that the models operate only on the input, which is uncorrelated with the noise (ref. 12).

| n/c, percent | Standard deviation in percent of nominal value | | | |
|---|---|---|---|---|
| | $\alpha_{oo}$ | $\alpha_{o1}$ | $\beta_{oo}$ | $\beta_{o1}$ |
| 7 | 6.3 | 3.3 | 8.7 | 4.1 |
| 14 | 12.7 | 6.7 | 16.8 | 8.0 |
| 28 | 22.6 | 13.6 | 32.4 | 15.2 |

Even though unbiased, the variance exhibited by the parameter estimates at the higher noise-to-signal ratios would make it extremely difficult, if not impossible, to determine the details of parameter variability if both noise and variability existed simultaneously.

Additive noise actually exists in the form of measurement inaccuracies in every experimental situation. The noise level from this source would normally be much lower than the maximum level shown in figure 8, but should it be of an unacceptably high level for any reason, some improvement in parameter estimate variance could be obtained by changing the makeup of the error vector. Referring to sketch (e), the error components would be changed to (with $m$ replacing $c$)

$$E_i = y_i - T_{i-1}m$$

where $i = 2, 3, \ldots$. The amount of improvement is dependent on the characteristics of the noise, increasing with increasing noise bandwidth with constant rms.

## CONCLUDING REMARKS

This report has considered the concept of applying integral transformations to the differential form of the system equation. This is a well-known concept that has received considerable attention as applied to parameter estimation because it transforms the system equation to a much more usable state variable form.

Two new developments have resulted from this study. The first is the extension of the concept to explicitly account for time-variable parameters. The second is the recognition of the viewpoint that, in addition to the traditional concept of providing a generalized structure for an equation error formulation of the parameter estimation problem, the transformed equations can also represent a generalized model structure for the output error formulation. This concept provides the output error formulation with stability properties which were not achievable in previous formulations.

The experimental results presented showed the ability of the new formulations to estimate parameters that are highly variable with time. They also show that extending the integral transformation concept to the output error formulation enables the use of this concept to obtain parameter estimates that are unbiased in the presence of noise introduced in the output.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., 94035, Jan. 6, 1969
125-19-01-16-00-21

## APPENDIX A

## DETAILS OF APPLYING THE INTEGRAL TRANSFORMATION

In this appendix, the essential details of carrying out the transformation equation (2) of the text will be illustrated. The transformation is carried out term by term on the sums of differentials enclosed by brackets. The first transformation of the zeroth derivative term is

$$I_1^o = \frac{1}{\tau}\int_o^t g_o(\xi)q(\xi)e^{\frac{\xi-t}{\tau}}\,d\xi \tag{A1}$$

where

$$q = c \text{ or } u$$

$$g = a \text{ or } b$$

and where the notation $I_j^k$ indicates the jth transformation of the kth derivative term. If equation (A1) is integrated by parts, the result is

$$I_1^o = g_o(t)T_1q - \frac{1}{\tau}\int_o^t \frac{dg_o}{d\xi}\,d\xi \int_o^\xi q(\xi_1)e^{\frac{\xi_1-t}{\tau}}\,d\xi_1 \tag{A2}$$

where $T_rq$ is defined by

$$T_rq = \frac{1}{\tau^r}\int_o^t e^{\frac{\xi_r-t}{\tau}}\,d\xi_r \;.\;.\;.\int_o^{\xi_3} e^{\frac{\xi_2-\xi_3}{\tau}}\,d\xi_2 \int_o^{\xi_2} q(\xi_1)e^{\frac{\xi_1-\xi_2}{\tau}}\,d\xi_1 \tag{A3}$$

Substituting the identity

$$t = \xi + (t - \xi)$$

in the exponent of equation (A2) gives

$$I_1^o = g_o(t)T_1q - \frac{1}{\tau}\int_o^t \frac{dg_o}{d\xi}e^{\frac{\xi-t}{\tau}}\,d\xi\int_o^\xi q(\xi_1)e^{\frac{\xi_1-\xi}{\tau}}\,d\xi_1$$

which, in the notation of equation (A3), can be written

$$I_1^o = g_o(t)T_1 q - \tau T_1(\dot{g}_o T_1 q) \tag{A4}$$

Upon integrating again, the result is

$$I_1^o = g_o(t)(T_1 q) - \tau \frac{dg_o}{dt}(T_2 q) + \frac{1}{\tau}\int_o^t \frac{d^2 g_o}{d\xi^2}\, d\xi \int_o^{\xi} e^{\frac{\xi_2 - t}{\tau}}\, d\xi_2 \int_o^{\xi_2} q(\xi_1) e^{\frac{\xi_1 - \xi_2}{\tau}}\, d\xi_1 \tag{A5}$$

Continuing this process then gives, in general,

$$I_1^o = g_o(t)(T_1 q) - \tau \frac{dg_o}{dt}(T_2 q) + \tau^2 \frac{d^2 g_o}{dt^2}(T_3 q) - \ldots \tag{A6}$$

The second transformation of the zeroth derivative term indicated in equation (2) is

$$I_2^o = \frac{1}{\tau}\int_o^t I_1^o\, e^{\frac{\xi - t}{\tau}}\, d\xi \tag{A7}$$

By proceeding as before, the result is found to be

$$I_2^o = g_o(t)(T_2 q) - 2\tau \frac{dg_o}{dt}(T_3 q) + 3\tau^2 \frac{d^2 g_o}{dt^2}(T_4 q) - \ldots \tag{A8}$$

and the general transformation of the zeroth derivative term is given by

$$I_j^o = \sum_{i=o}^{\infty} (-1)^i \frac{(i+j-1)!}{i!(j-1)!} \tau^i \frac{d^i g_o}{dt^i}(T_{i+j} q) \tag{A9}$$

The next transformation in equation (2) to be considered is that of the first derivative term:

$$I_1^1 = \frac{1}{\tau}\int_o^t g_1(\xi) \frac{dq}{d\xi} e^{\frac{\xi - t}{\tau}}\, d\xi \tag{A10}$$

Integrating by parts gives

$$I_1^1 = \frac{1}{\tau} g_1(t)q(t) - \frac{1}{\tau} g_1(0)q(0)e^{-\frac{t}{\tau}} - \frac{1}{\tau^2}\int_0^t g_1(\xi)q(\xi)e^{\frac{\xi-t}{\tau}} d\xi - \frac{1}{\tau}\int_0^t \frac{dg_1}{d\xi} q(\xi)e^{\frac{\xi-t}{\tau}} d\xi \tag{A11}$$

The first integral is seen to be the same as in equation (A1) with $g_1$ replacing $g_o$, and likewise for the second integral, with the derivative of $g_1$ replacing $g_o$. Then the first transformation of the first derivative term is

$$I_1^1 = \frac{1}{\tau} g_1(t)(q-T_1q) - \frac{dg_1}{dt}(T_1q-T_2q) + \tau\frac{d^2g_1}{dt^2}(T_2q-T_3q) \; . \; . \; . - \frac{1}{\tau} g_1(0)q(0)e^{-t/\tau} \tag{A12}$$

The second transformation of the first derivative term is

$$I_2^1 = \frac{1}{\tau}\int_0^t I_1^1 e^{\frac{\xi-t}{\tau}} d\xi \tag{A13}$$

which can also be expressed in terms of lower order transformations to obtain

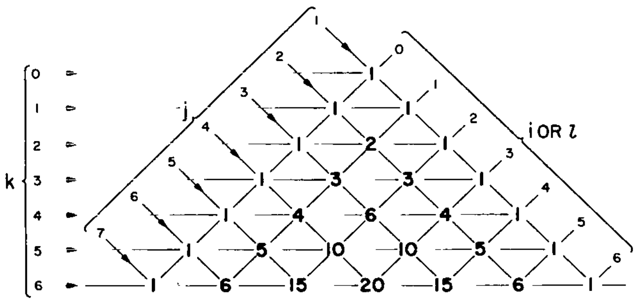
$$I_2^1 = \frac{1}{\tau} g_1(t)(T_1q-T_2q) - 2\frac{dg_1}{dt}(T_2q-T_3q) + 3\tau\frac{d^2g_1}{dt^2}(T_3q-T_4q) \ldots - \frac{1}{\tau} g_1(0)q(0)\left(\frac{t}{\tau}\right) e^{-\frac{t}{\tau}} \tag{A14}$$

and, in general,

$$I_j^1 = \sum_{i=o}^{\infty} (-1)^i \frac{(i+j-1)!}{i!(j-1)!} \tau^{i-1} \frac{d^ig_1}{dt^i}\left[T_{i+j-1}q - T_{i+j}q\right] - \tau^{-1}g_1(0)q(0)\left(\frac{t}{\tau}\right)^{j-1} e^{-t/\tau} \tag{A15}$$

The same integration pattern holds for the transformation of all higher derivative terms in equation (2), thus leading to the general term given by equation (5).

The numerical coefficients $(i+j-1)!/i!(j-1)!$ in equation (5) will be recognized as the values along the $j$th diagonal of Pascal's triangle, and the coefficients $k!/l!(k-l)!$ will be recognized as the values along the $k$th




Sketch (j)

row. This is a simple but effective aid to writing the transformed system equations and is illustrated in sketch (j).

The transformation term $f(t)$ in equation (5) generated by the initial conditions will not be developed in general. It is composed of initial values of the state variables and the parameters, and initial derivatives of both, up to and including the (k-1)th derivative, and decays to zero due to the exponential term as shown in equation (A15).

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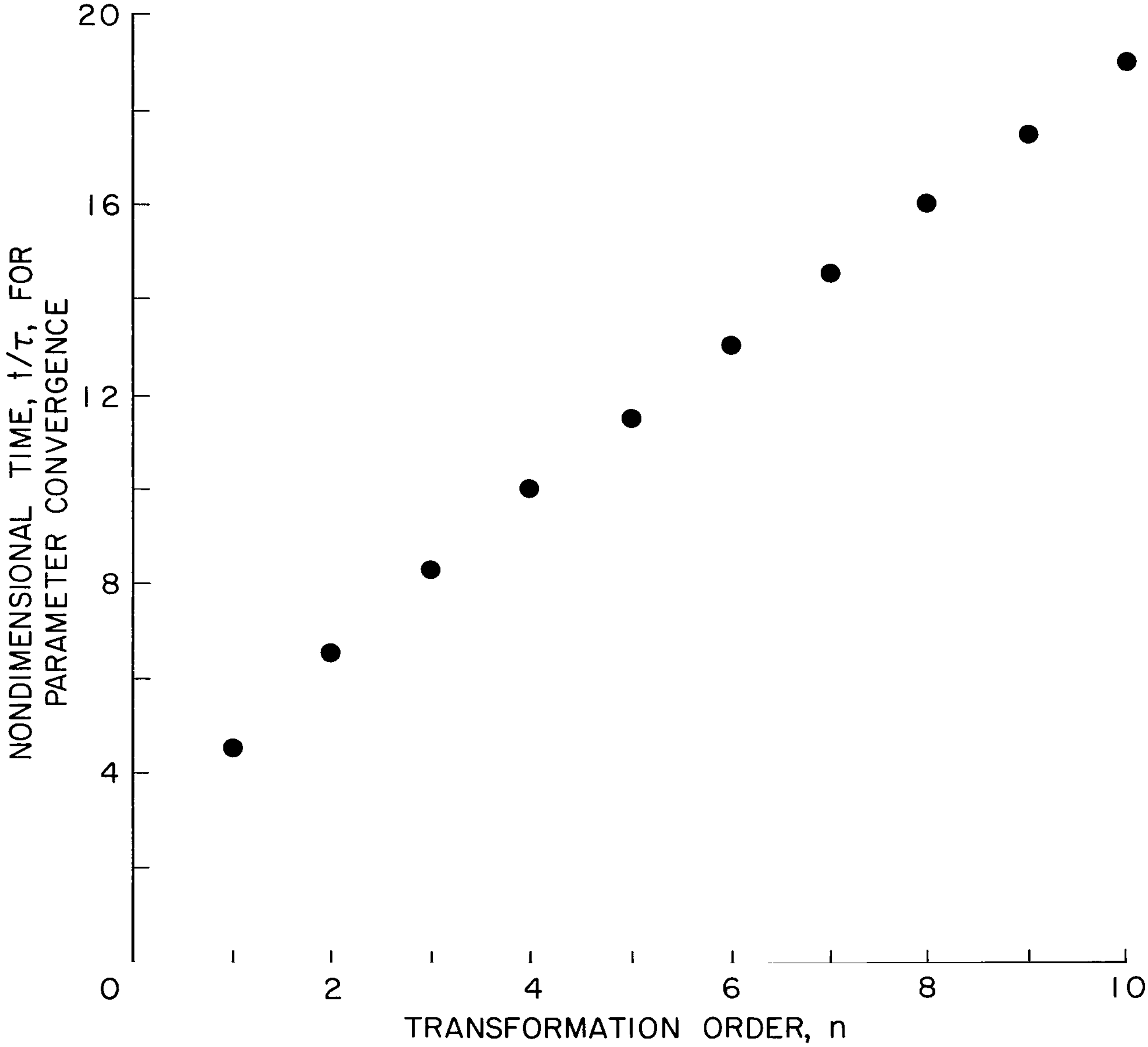
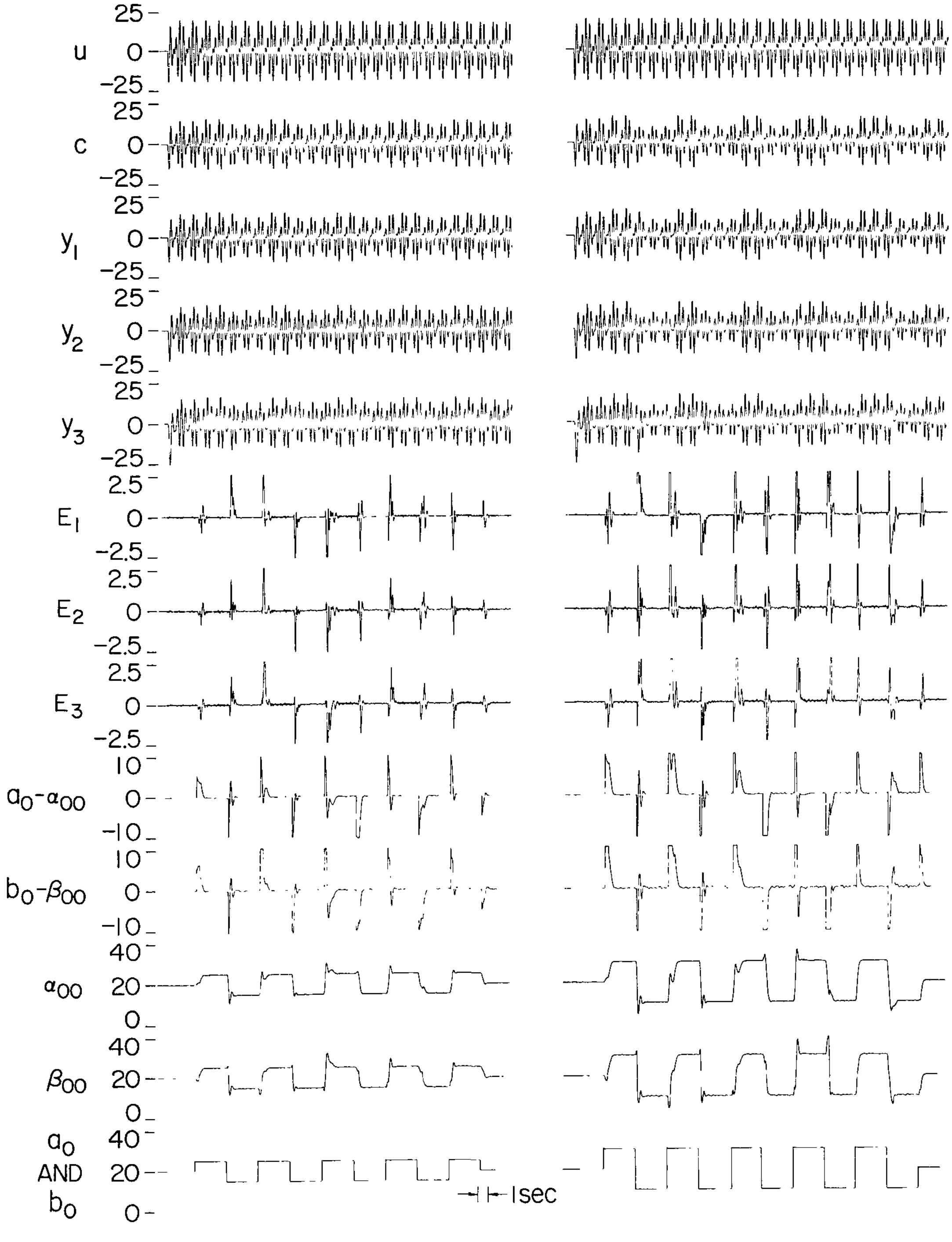
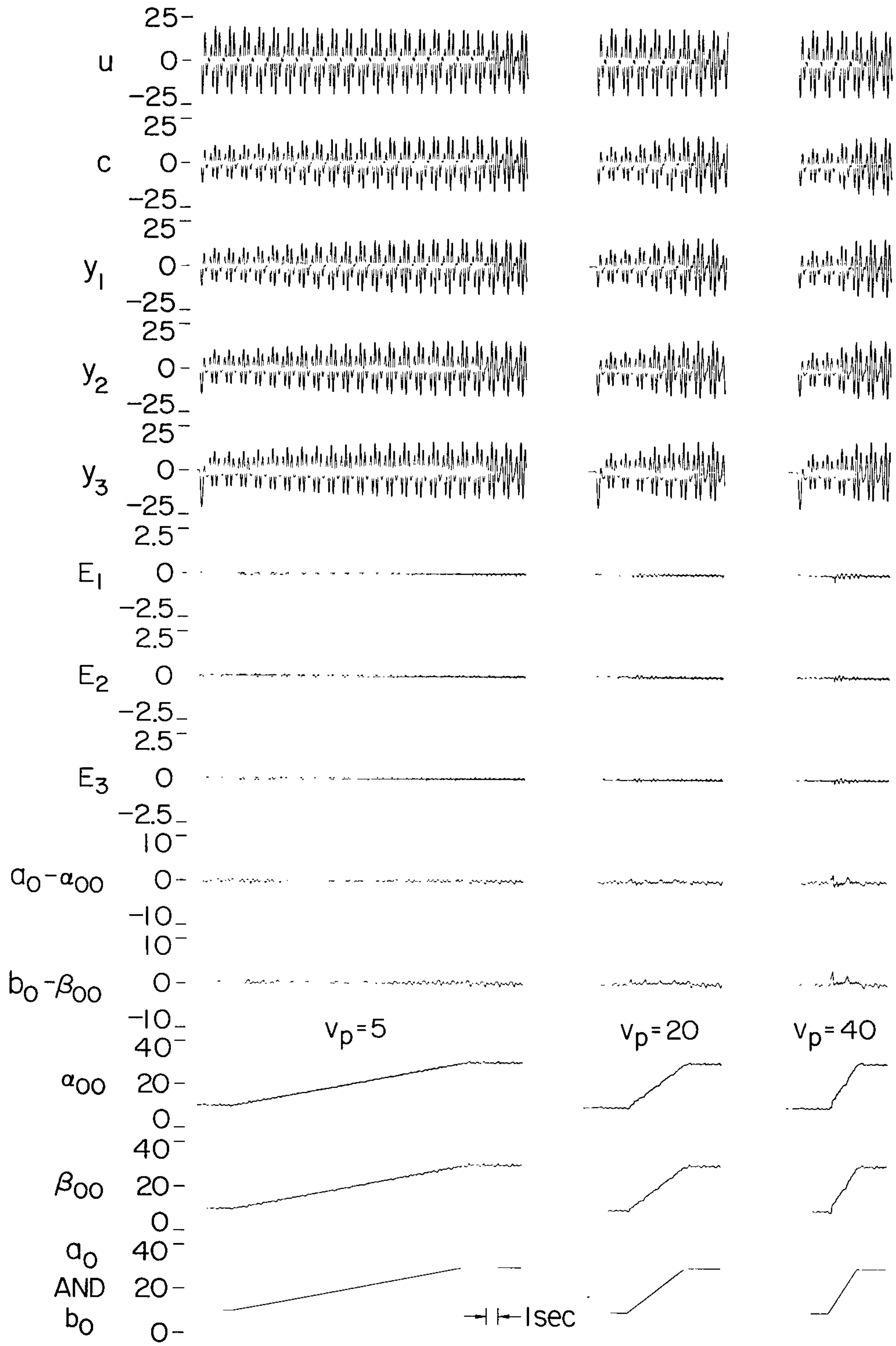



Figure 1.- Time for parameter convergence; output error formulation.



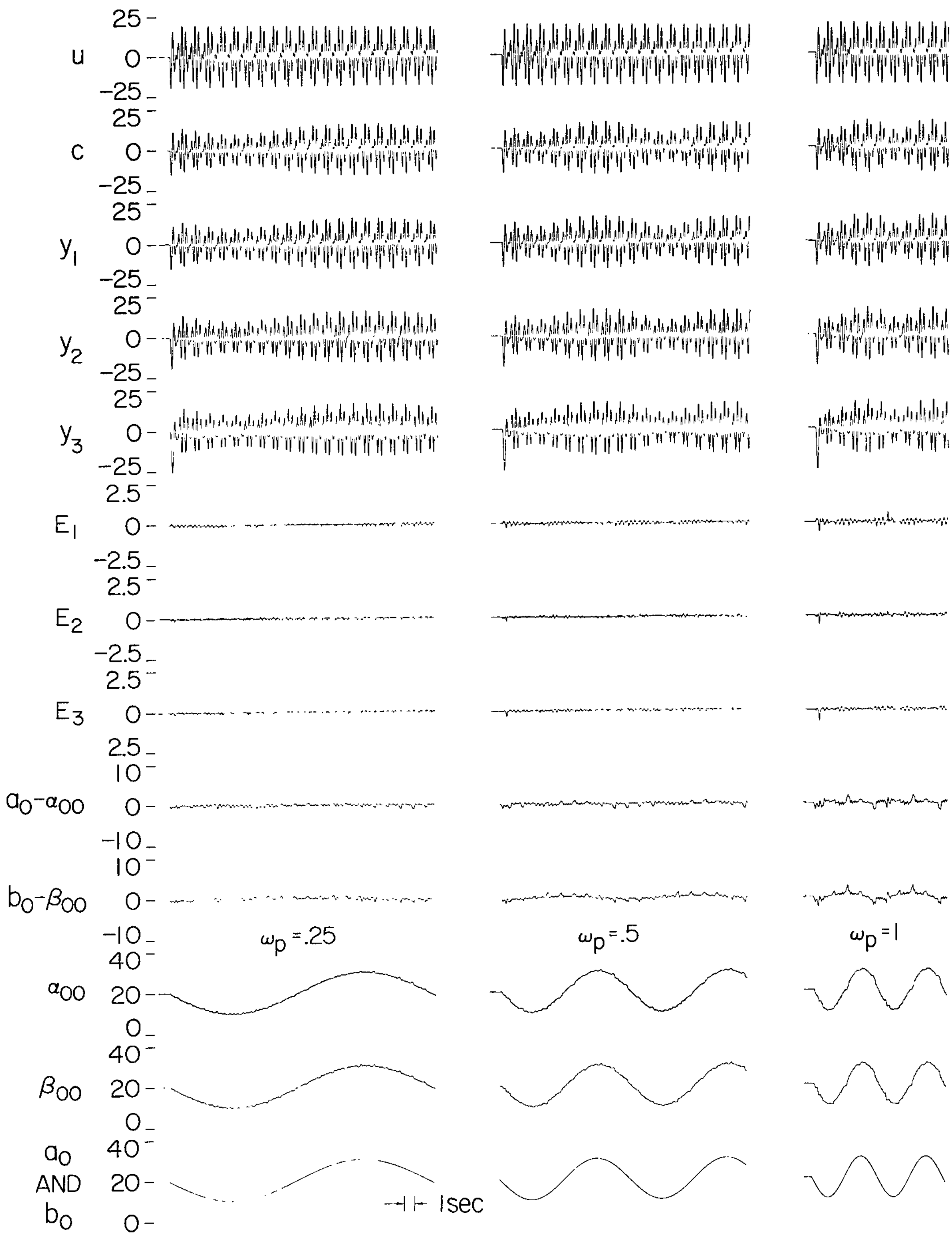

(a) Step variation.

Figure 2.- Estimation of time variable parameters; first-order system.



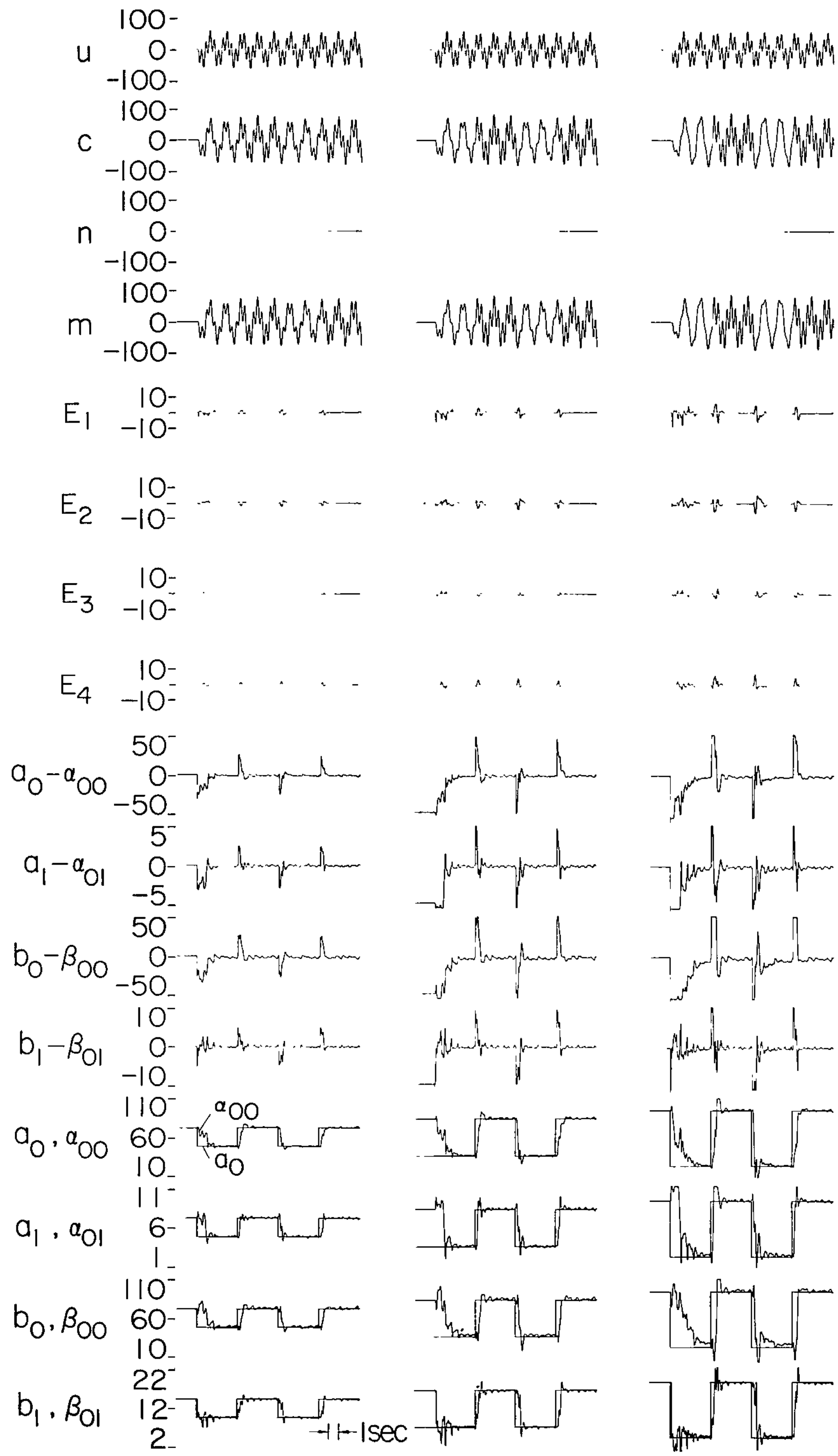

(b) Ramp variation.

Figure 2.- Continued.



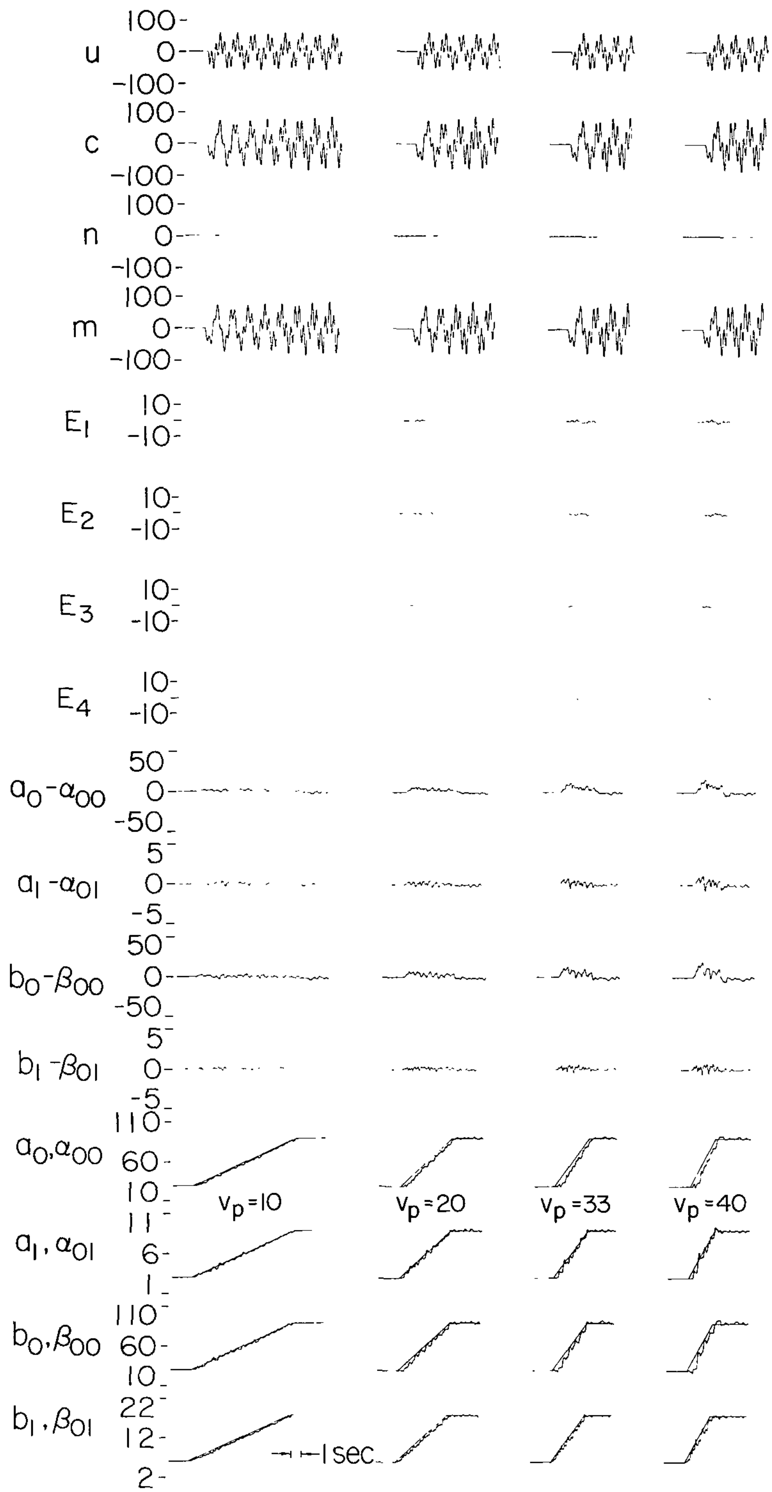

(c) Sinusoidal variation.

Figure 2.- Concluded.



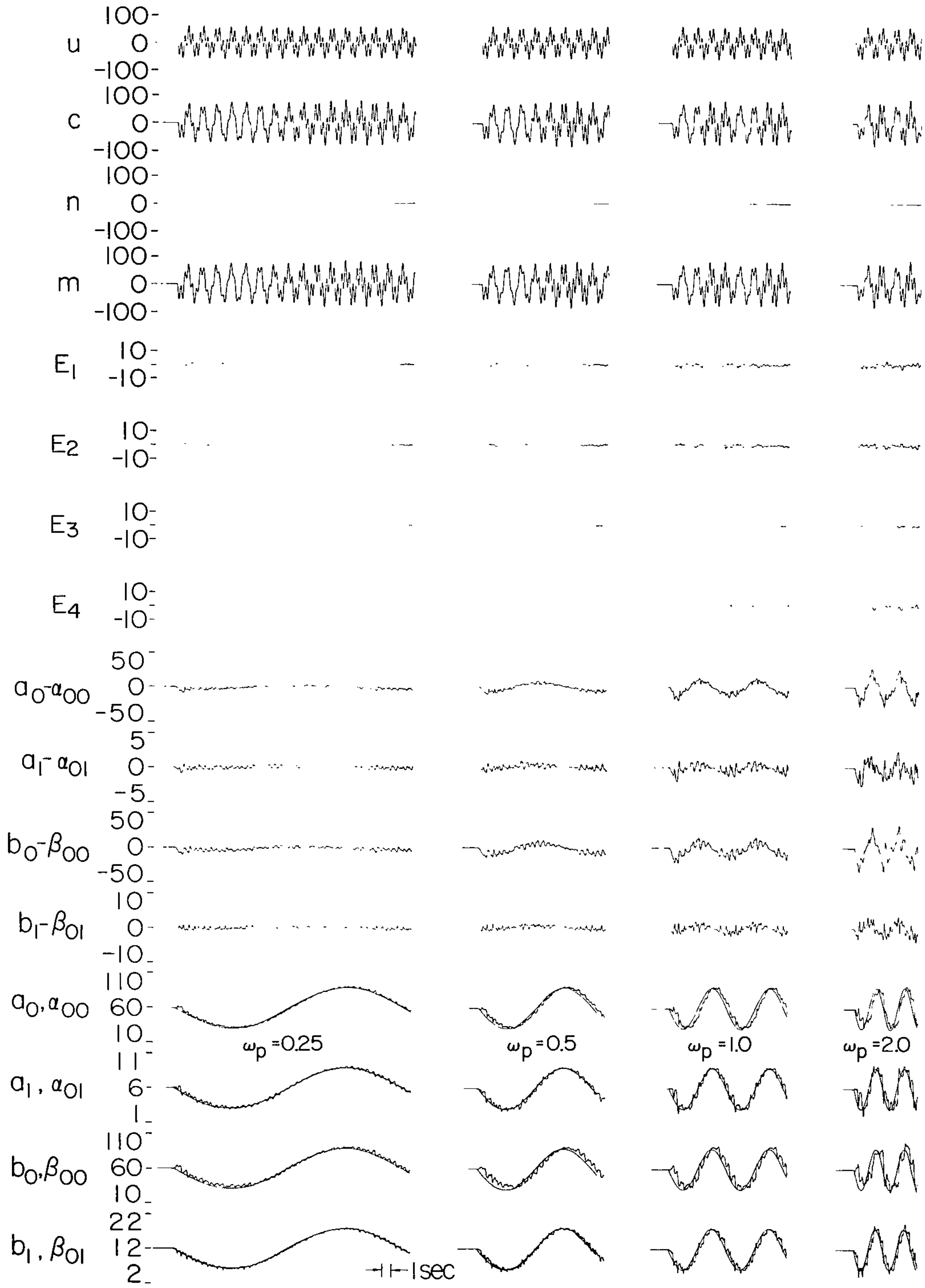

(a) Step variation.

Figure 3.- Estimation of time variable parameters; second-order system.




(b) Ramp variation.

Figure 3.- Continued.




(c) Sinusoidal variation.

Figure 3.- Concluded.

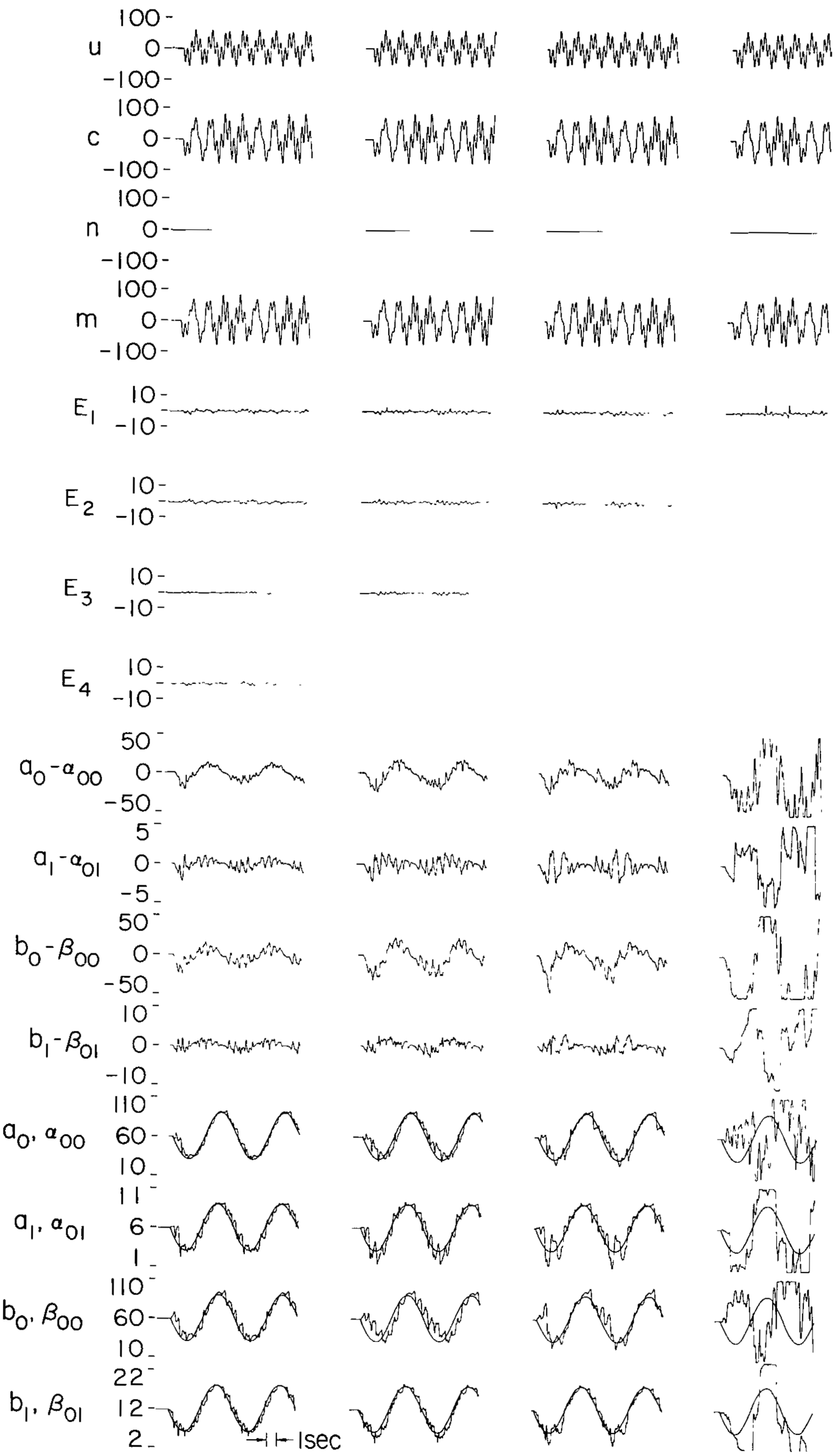



Figure 4.- Effect of number of error components on the estimation of sinusoidally varying parameters; second-order system, $\omega_p = 1$.

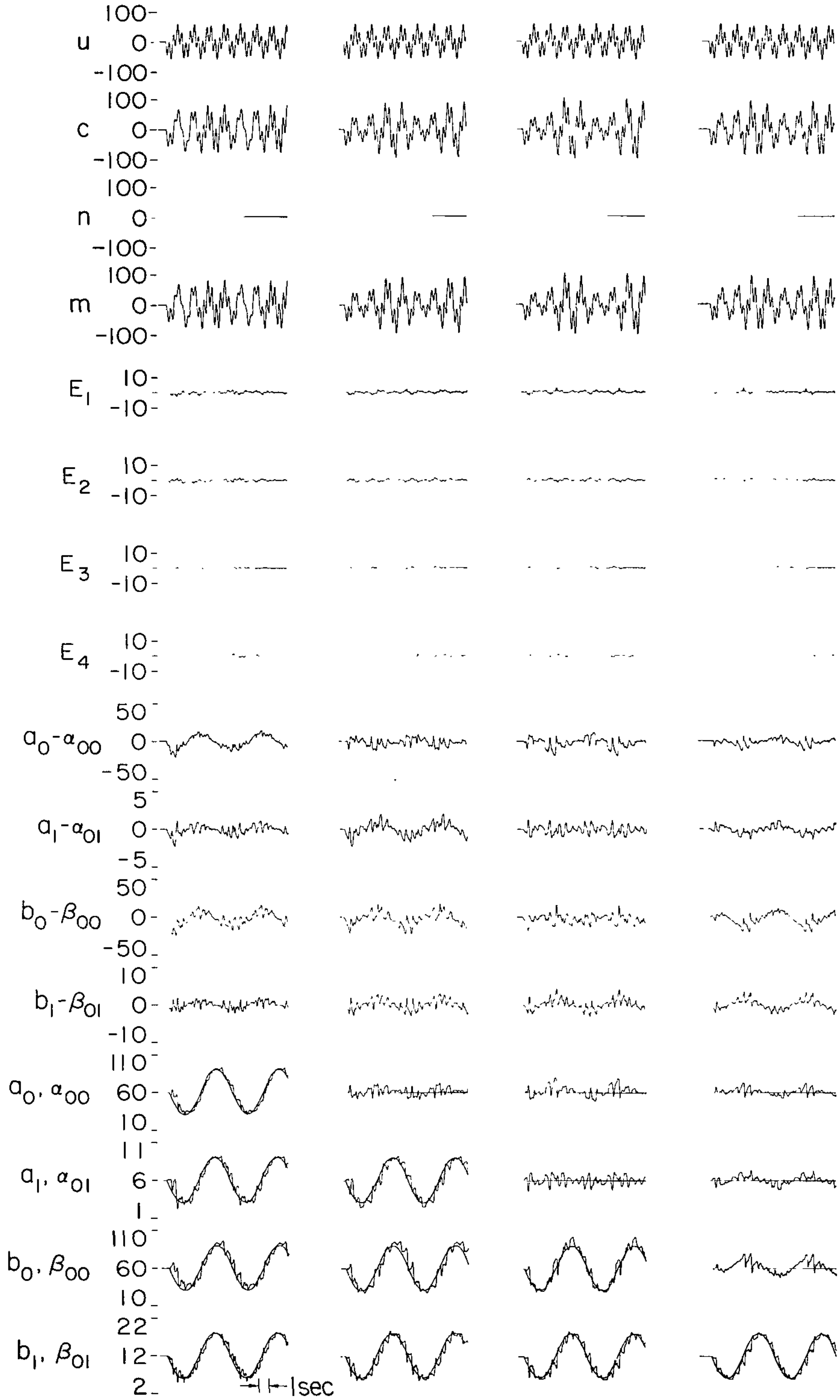



Figure 5.- Effect of number of varying parameters on the accuracy of estimation; second-order system, $\omega_p = 1$.

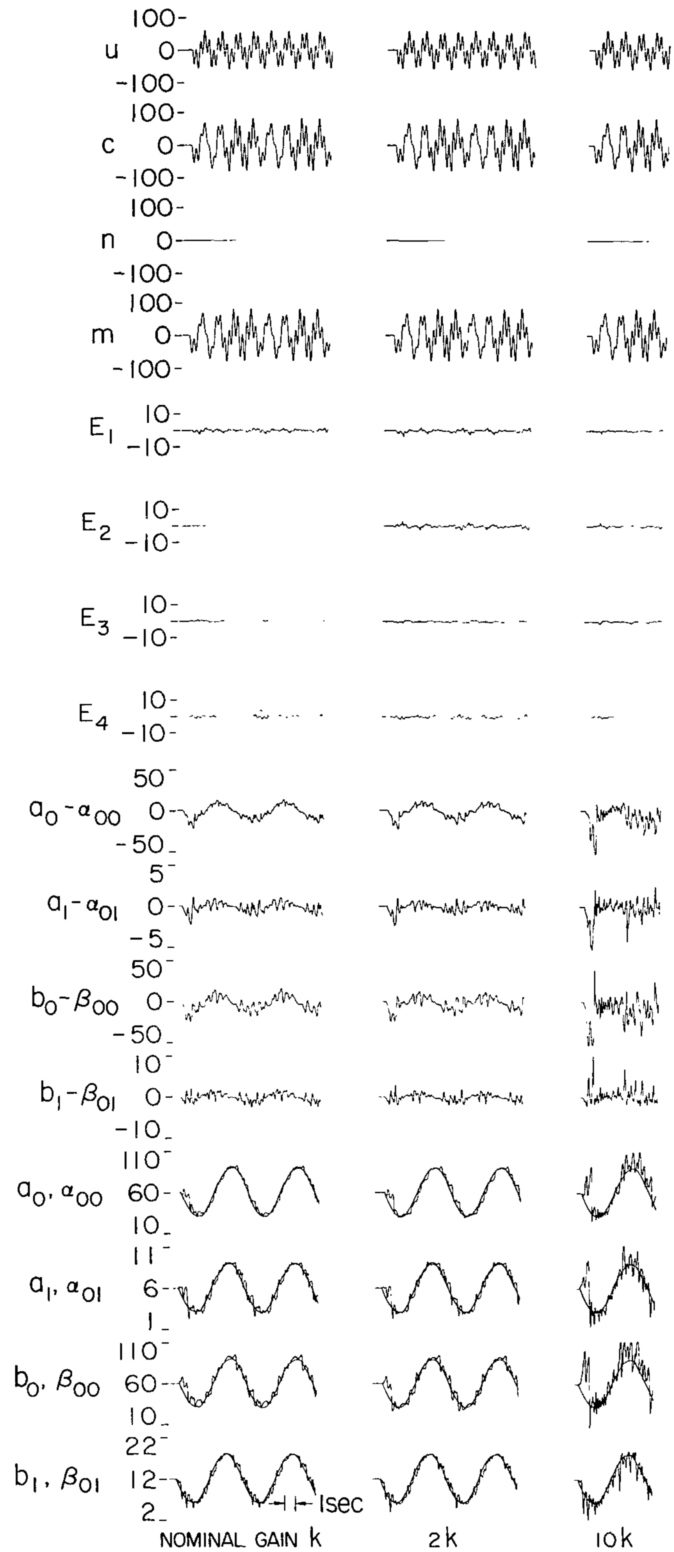



Figure 6.- Effect of adjustment gain on the accuracy of parameter estimation; second-order system, $\omega_p = 1$.

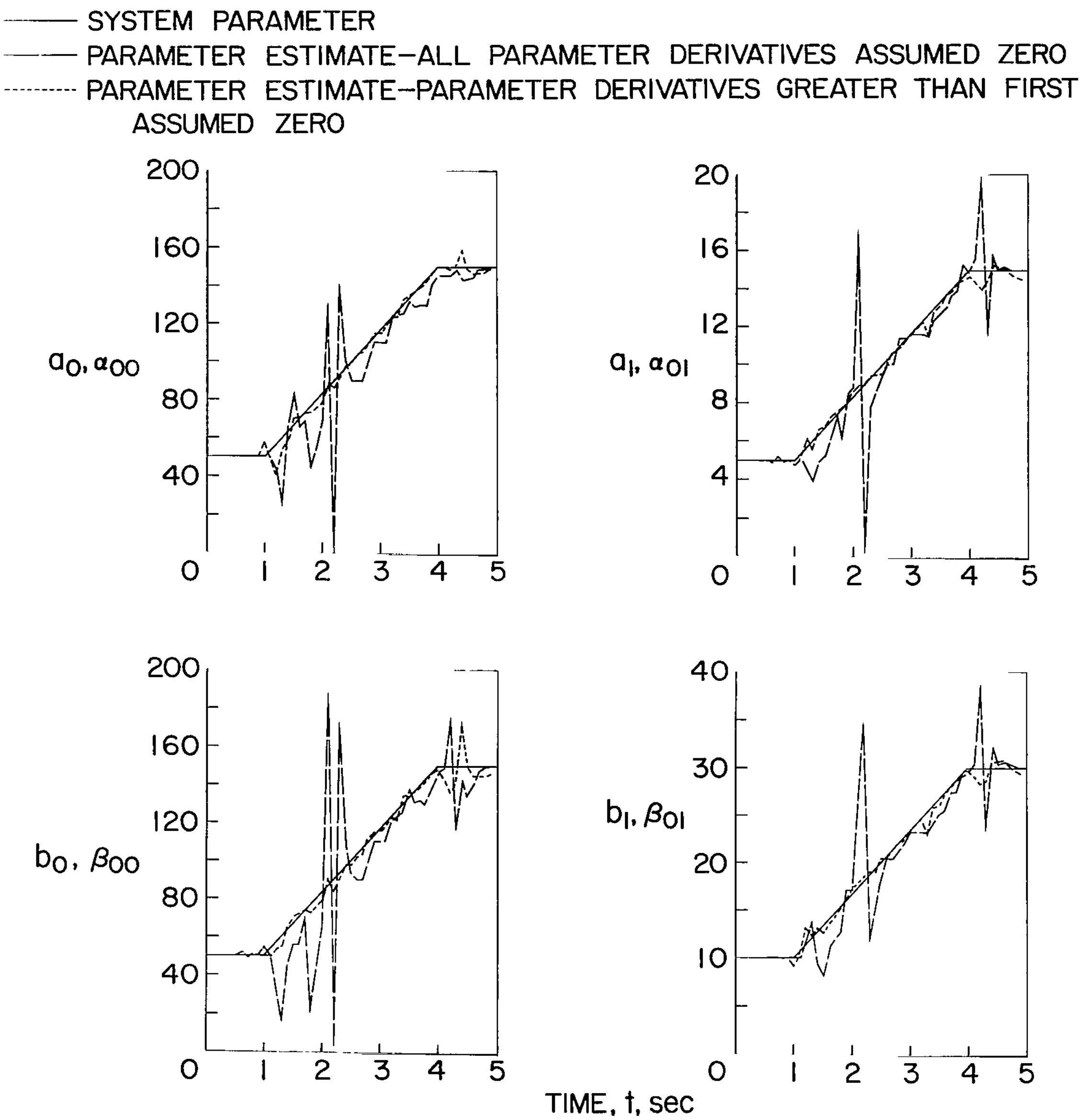



Figure 7.- Estimation of time variable parameters by digital mechanization of the equation error formulation; second-order system.

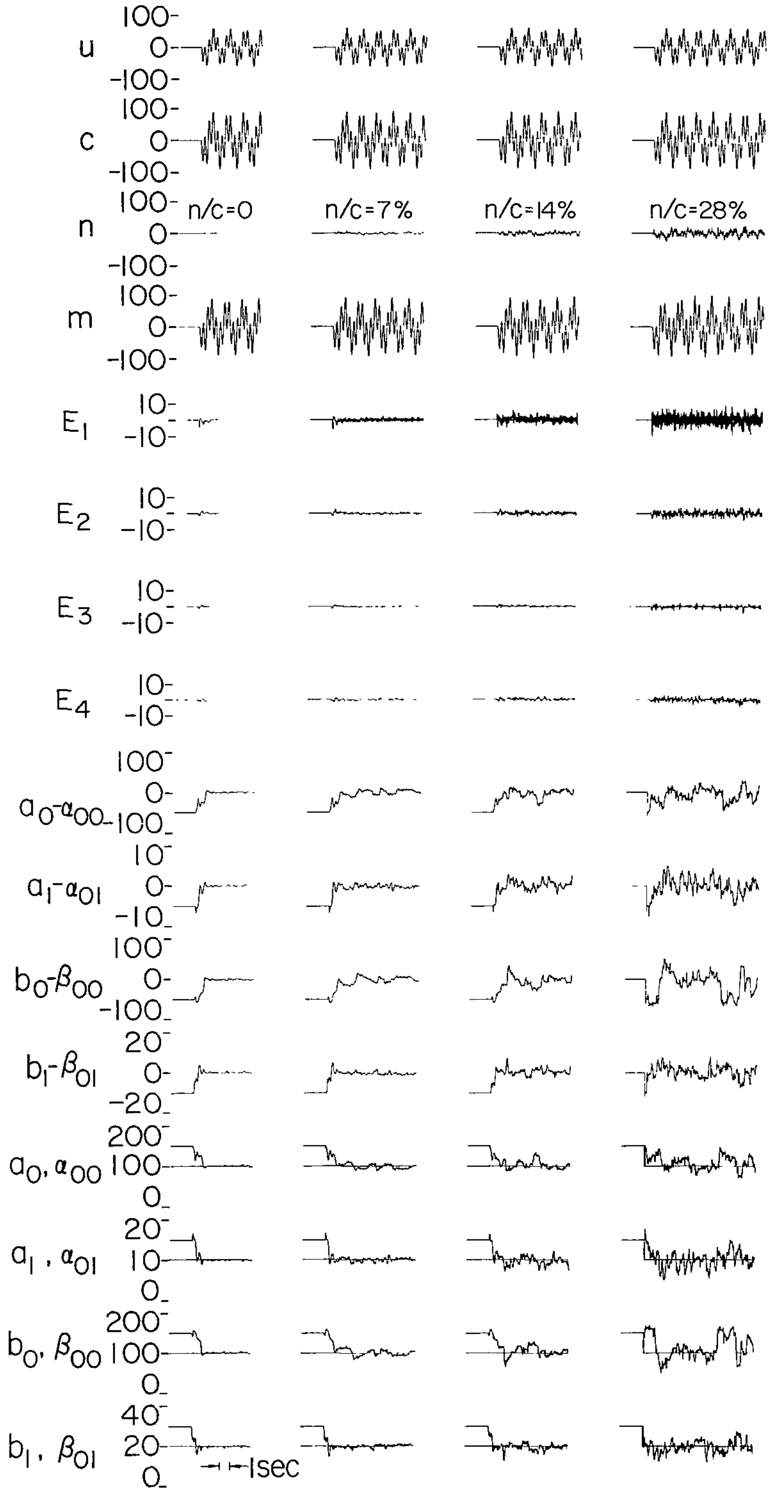



Figure 8.- Effects of noise on parameter estimation; second-order system.



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